A Long-Term Mathematical Model for Mining Industries

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1 Introduction

Mean field type models describing the asymptotic behavior of stochastic differential games (Nash equilibria) as the number of players tends to $+\infty$ have been introduced and termed *mean field games* in [7, 8, 9]. For brevity, the acronym *MFG* will sometimes be used for *mean field games*. Since its introduction ten years ago, the topic has attracted the attention of many researchers, so many that it has become almost impossible to give all the references. The models have been applied to many areas such as economics, finance, social sciences and engineering. Examples of MFG models with applications in economics and social sciences are proposed in [5, 1]. The first articles [7, 8, 9] mostly dealt with the cases when each player is exposed to an independent source of risk (idiosynchratic risk). The case when there is a risk common to all players (maybe in addition to the abovementionned idiosynchratic risks) is much more difficult and has been first discussed in [10]. Such models lead to the so-called *master equations*, a term chosen for some second order partial differential equations set in a space of probability measures. The well-posedness of master equations have been discussed in the recent article [4] in some particular cases, together with the convergence as the number of players tends to $+\infty$. When the set of states is finite, MFG models may lead to systems of hyperbolic equations: this will be precisely the case in the specific model discussed hereafter.

Mining industries have several specifities which are well taken into account by mean field games (MFG) models. The present work is devoted to the dynamics of mining industries on very long time periods and at an aggregate level, for which a MFG model will be proposed.
More precisely, we are interested in steady state MFG systems that lead to a good approximation of the so called *cost curve* and involve a quite parcimonious parametrization. The interest of this reduced parametric model is to lead to low dimensional MFG systems that can be solved numerically and tested by comparison with available historical data. The adequation of the model with historical observations a posteriori supports its validity.
The interest of the models proposed in the present paper in terms of economical analysis will be tackled in more details in another article in preparation, in which we shall analyze the qualitative and quantitative agreement of the model with the economical observations and the historical data. Therefore, after a short introduction on mining industries, we will focus on the mathematical and numerical aspects of our models and their calibration from historical data, and leave most of the economical interpretation for the other paper.

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In mining industries, the capital has essentially two functions: prospection for new deposits, and construction of new extraction facilities, which create a production system characterized by: the available reserve of ore, the annual production capacity, the operational cost of extraction. Production decreases the available reserve, and ceases when the reserve is drained. On the long term, mining industries rely on a continuous flux of investments on prospection and building extraction capacities.
The present model is adapted to the long-term analysis (on several decades) of the dynamics of the prices of commodities on global markets. We shall neglect short-term phenomenons and regional disparities.
Mining producers are distinguished from each other by their prospection costs, their costs for constructing facilities, and their operational costs of extraction. We shall however simplify the model by supposing that the costs of prospection and of building facilities only depend on the size of the available reserve possessed by the producer. We shall also assume that the operational cost of production are constant in time and can take only a small number n of different values. In the language of mining industry, this amounts to saying that the *cost curve* is piecewise constant. We shall see that the simplest model, namely with $n = 1$, is already in good adequation with the historical data on four decades.
On the mathematical and numerical viewpoint, the model leads to a MFG system in n dimensions, with specific difficulties related in particular to boundary conditions, and which is equivalent to a Hamilton-Jacobi-Bellman equation in the absence of economical frictions.

2 Mathematical models

2.1 The simpler model: a closed industry with a single technology

2.1.1 The main economical assumptions

We consider a mining industry composed of production units in competition. In this first model, only the existing production units can prospect for new resource and build new production facilities.
The main assumptions are as follows:

- The term reserves means *reserves immediately available for production.*
- Each production unit has two controls: the production rate and the investments in prospecting for new reserves/building new production structures. In what follows, we will not make the distinction between prospecting for new reserves and building new production facilities. Both activities will be termed *prospection.*
- There is only one type of production unit, i.e. there is only one technology; thus all units have the same prospection and production costs.
- Scale invariance: there are no nonlinear effects, i.e. the production of a production unit is proportional to its reserve. This assumption makes it possible to consider that all production units have the same unitary reserve (this amounts to splitting production units so that they all have unitary reserve)

More precisely,

- The total reserve will be noted $R(t) \in \mathbb{R}_+$. With the convention above, $R(t)$ can also be viewed as the quantity of production units.

- It is assumed that the production capacity is proportional to the reserve. Let k, $0 < k \leq 1$ be the production capacity of a single production unit (with reserve 1).
- Let $c > 0$ be the unitary production cost, i.e. the production cost of a unit of ore.
- Each production unit can invest into prospection. The flux invested into prospection by a single unit is αdt (α is a control parameter). An investment rate of α results in increasing the reserves at a rate $\phi(\alpha)$, where ϕ is an increasing and concave function on $\mathbb{R}_+$ such that $\phi(0) = 0$. For example, one may choose $\phi(\alpha) = \sqrt{\alpha}$. Translated into the equivalent model where production units may have different sizes, this means that if the reserve of a production unit is ρ, and the flux spent into investment during dt is z, then the reserve will be increased by $\rho\phi(z/\rho)dt$.
- The other control parameter is the production rate β of a single unit, with $0 \leq \beta \leq k$.
- The discount factor of the expected income is $r > 0$.

Systemic risk: an exogeneous demand function A source of common noise is the exogeneous demand function: it is assumed that the demand function has the form $D(X, p) = X\tilde{D}(p)$ where p is the unitary price of ore and X is a random positive parameter standing for the state of the economy. The function $\tilde{D}$ is defined on $(0, +\infty)$, takes nonnegative values, is nonincreasing with respect to $p > 0$ and decreasing in the interval $\tilde{D}^{-1}(0, +\infty)$. It tends to 0 at $+\infty$. For example, one can take $\tilde{D}(p) = (1 - \epsilon p)_+$ where ϵ is a positive parameter, or $\tilde{D}(p) = p^{-s}$ where s is a positive exponent.
The dynamics of X is assumed to be of the form

$$dX_t = X_t(bdt + \sigma dW_t), \tag{1}$$

where W_t is a standard Brownian motion and b is the average growth rate of the economy, and the volatility σ is a nonnegative constant (for simplicity).

2.1.2 The strategy of the production units

Let $u(R, X)$ be the expected benefit of a production unit, discounted by r, or equivalently the value of a production unit. When a production unit produces q units of ore, its costs can be shared into two parts:

- a production cost of qc
- a decrease of the reserves of q, which costs $qu(R, X)$,

so the total cost is $qc + qu(R, X)$ while the income is pq. Therefore, the following inequality must hold:

$$p \geq c + u(R, X). \tag{2}$$

If $p = c + u(R, X)$, it is indifferent for a production unit to produce or not.

Fixing the price p and the global production Q knowing $u(R, X)$ The unit price p of ore and the global production Q can be found by using (2) and matching offer and demand.
Let $P^*(R, X, u)$ and $Q^*(R, X, u)$ be respectively the price and global production functions. The cash income for a unit of ore produced by the industry is

$$g(R, X, u) = P^*(R, X, u) - c. \tag{3}$$

There are thus two cases:

1. **The industry produces at full capacity** when $p > u + c$.
   The total production is $Q = kR$. Matching offer and demand yields $kR = D(X, p)$, i.e $P^*(R, X, u) = \tilde{D}^{-1}(kR/X)$. The inequality $P^*(R, X, u) > u + c$ is then equivalent to $\tilde{D}(u + c) > kR/X$. In this regime, $g(R, X, u) = \tilde{D}^{-1}(kR/X) - c$.

2. **The industry has a partial production** when $u + c = p$. The total production Q is obtained by matching offer and demand: $Q = D(X, u+c)$. In this regime, $0 \leq \tilde{D}(u+c) < kR/X$ and $g(R, X, u) = u$.

To summarize, setting $\tilde{D}^{-1}(z) = -\infty$ if $z > \lim_{p \to 0^+} \tilde{D}(p)$, and $\tilde{D}(p) = +\infty$ is $p < 0$,

$$P^*(R, X, u) = \max\left(\tilde{D}^{-1}\left(\frac{kR}{X}\right), u + c\right), \tag{4}$$

$$\frac{Q^*(R, X, u)}{X} = \min\left(\tilde{D}(u + c), \frac{kR}{X}\right), \tag{5}$$

$$g(R, X, u) = \max\left(\tilde{D}^{-1}\left(\frac{kR}{X}\right) - c, u\right). \tag{6}$$

As always in mean field games, we are interested in finding an equilibrium where

1. each production unit chooses its strategy given the dynamics of some aggregate quantities, here R_t (and X_t which is exogeneous)

2. conversely, the evolution of the aggregate quantity R_t is deduced from the previously mentioned individual optimal controls.

The optimal strategy of a production unit The value u is obtained by optimizing on the controls, knowing the dynamics of the aggregate quantities R and X

$$u(R, X) = (1 - rdt) \max_{\alpha > 0, 0 \leq \beta \leq k} \mathbb{E}\left(\begin{array}{l} (\beta g(R, X, u) - \alpha)dt \\ +(1 + \phi(\alpha)dt - \beta dt)u(R + dR, X + dX) \end{array} \right).$$

A first order expansion yields

$$\begin{aligned} 0 = & - ru(R, X) + k(g(R, X, u) - u(R, X)) + \max_{\alpha} (\phi(\alpha)u(R, X) - \alpha) \\ & + \partial_R u \frac{dR}{dt} + \left(bX\partial_X u + \frac{1}{2}\sigma^2 X^2 \partial_{XX} u\right), \end{aligned} \tag{7}$$

where the optimal β has been given by

$$\beta^* = k \qquad \text{if } g(R, X, u) - u > 0, \tag{8}$$

$$\beta^* = \frac{D(X, u + c)}{R} \qquad \text{if } g(R, X, u) - u = 0. \tag{9}$$

While (8) is unambiguous, (9) needs to be explained: in the regime when $g(R, X, u) - u = 0$, the units are indifferent to producing or not, hence, β^* is not really characterized. To fix β^*, we have arbitrarily imposed that all units behave the same way, so in (9), β^* is obtained by matching the global offer and the demand: $R\beta^* = D(X, u + c)$. This choice does not affect (7), because β is multiplied by $g(R, X, u) - u = 0$.

At the equilibrium, if α^* is the optimal value of α and if $Q^*(R, X, u)$ is the demand $Q^*(R, X, u) = R\beta^*$, then the aggregate quantity R evolves as follows:

$$dR = (R\phi(\alpha^*) - Q^*(R, X, u))\, dt. \tag{10}$$

To summarize, we get the partial differential equation

$$-ru+k\left(g(\cdot,u)-u\right)-Q^*(\cdot,u)\partial_R u+\partial_R\left(R\max_\alpha(u\phi(\alpha)-\alpha)\right)+bX\partial_X u+\frac{1}{2}\sigma^2X^2\partial_{XX}u=0. \quad (11)$$

We are interested in nonnegative solutions of (11).

2.1.3 Reduced forms of (11)

Homogeneity: a reduced variable Observe that g and $\tilde{Q}^*=Q^*/X$ only depend on R/X and u. Introduce the reduced variable $y=R/X$; looking for a nonnegative solution of (11) of the form $u(R,X)=v(y)$, we obtain the second order differential equation:

$$-rv+k(g(y,v)-v)-\tilde{Q}^*(y,v)v'+\frac{d}{dy}\left(y\max_\alpha(v\phi(\alpha)-\alpha)\right)+(\sigma^2-b)yv'+\frac{\sigma^2y^2}{2}v''=0. \quad (12)$$

Using (5) and (6), (12) becomes

$$\begin{aligned}0=&-rv+k1_{\{\tilde{D}^{-1}(ky)\geq v+c\}}\left(\tilde{D}^{-1}(ky)-c-v-yv'\right)-1_{\{\tilde{D}(v+c)<ky\}}\tilde{D}(v+c)v'\\&+\frac{d}{dy}\left(y\max_\alpha(v\phi(\alpha)-\alpha)\right)+(\sigma^2-b)yv'+\frac{\sigma^2y^2}{2}v''.\end{aligned} \quad (13)$$

We are interested in nonnegative solutions of (13).

A Hamilton-Jacobi equation For what follows, it is useful to notice that for any $M>0$, the function defined on $\mathbb{R}^+\times\mathbb{R}_+$ by

$$y\mapsto 1_{\{\tilde{D}(v+c)<ky\}}\int_{v+c}^M\tilde{D}(z)dz-1_{\{\tilde{D}(v+c)\geq ky\}}\left(ky\left(v+c-\tilde{D}^{-1}(ky)\right)-\int_{D^{-1}(ky)}^M\tilde{D}(z)dz\right)$$

is a primitive of

$$y\mapsto k1_{\{\tilde{D}^{-1}(ky)\geq v+c\}}\left(\tilde{D}^{-1}(ky)-c-v-yv'\right)-1_{\{\tilde{D}(v+c)<ky\}}\tilde{D}(v+c)v'.$$

Let the Hamiltonian $H_1(y,v)$ be the continuous function defined on $\mathbb{R}_+\times\mathbb{R}_+$ by

$$\begin{aligned}&H_1(y,v)\\&=-1_{\{\tilde{D}(v+c)<ky\}}\int_{v+c}^M\tilde{D}(z)dz+1_{\{\tilde{D}(v+c)\geq ky\}}\left(ky\left(v+c-\tilde{D}^{-1}(ky)\right)-\int_{D^{-1}(ky)}^M\tilde{D}(z)dz\right).\end{aligned} \quad (14)$$

Note the following monotonicity property:

$$(H_{1,v}(y,v)-H_{1,v}(z,w),v-w)-(H_{1,y}(y,v)-H_{1,y}(z,w),y-z)\leq 0. \quad (15)$$

It is also useful to introduce the second Hamiltonian

$$H_2(y,v)=-y\max_{\alpha\geq 0}(v\phi(\alpha)-\alpha), \quad (16)$$

which also satisfies (15). Finally, let the global Hamiltonian $H(y,v)$ be defined by

$$H(y,v)=H_1(y,v)+H_2(y,v). \quad (17)$$

Then (13) admits the conservative form

$$(b-r)v - \frac{d}{dy}\left(H(y,v)\right) + \frac{d}{dy}\left(\frac{\sigma^2 y^2}{2}\frac{dv}{dy} - byv\right) = 0.$$

Consider now the Hamilton-Jacobi equation

$$(r-b)V + byV' + H(y,V') - \frac{\sigma^2 y^2}{2}V'' = 0. \tag{18}$$

By deriving (18), we observe that if V is a nondecreasing solution to (18), then $v = V'$ is a nonnegative solution to (12).

2.1.4 Two special cases

The demand law is of the form

$$D(X,p) = X(1-\epsilon p)_+.$$

In this case, (11) becomes

$$\begin{aligned}0 = &- ru + k 1_{\{kR\le X\}}\left(\frac{1-\frac{kR}{X}}{\epsilon} - c - u\right)_+ - \min\left((1-\epsilon(u+c))_+, \frac{kR}{X}\right) X\partial_R u \\ &+ \partial_R\left(R\max_\alpha(u\phi(\alpha)-\alpha)\right) + bX\partial_X u + \frac{1}{2}\sigma^2 X^2 \partial_{XX} u,\end{aligned}$$

and the reduced equation(13) becomes

$$\begin{aligned}0 = &- rv + \frac{k1_{\{ky\le 1\}}}{\epsilon}(1-\epsilon(v+c)-ky)_+ - kyv' + ((1-\epsilon(v+c))_+ - ky)_-\, v' \\ &+ \frac{d}{dy}\left(y\max_\alpha(v\phi(\alpha)-\alpha)\right) + (\sigma^2 - b)yv' + \frac{\sigma^2 y^2}{2}v''.\end{aligned}$$

In (18), H is given by (17) and H_1 is defined on $\mathbb{R}_+\times\mathbb{R}_+$ by

$$\begin{aligned}H_1(y,v) &= -1_{\{1-\epsilon(v+c)<ky\}}\frac{(1-\epsilon(v+c))^2}{2\epsilon} + 1_{\{1-\epsilon(v+c)\ge ky\}}\frac{ky}{\epsilon}\left(\epsilon(v+c)-1+\frac{ky}{2}\right) \\ &= \frac{ky}{\epsilon}\left(\epsilon(v+c)-1+\frac{ky}{2}\right) - \frac{(\epsilon(v+c)-1+ky)_+^2}{2\epsilon},\end{aligned}$$

which is obtained by choosing $M = \frac{1}{\epsilon}$ in (14). Then (18) becomes

$$(r-b)V + (b+k)yV' + H_2(y,V') + \frac{ky}{\epsilon}\left(\epsilon c - 1 + \frac{ky}{2}\right) - \frac{(\epsilon(V'+c)-1+ky)_+^2}{2\epsilon} - \frac{\sigma^2 y^2}{2}V'' = 0. \tag{19}$$

If $\phi(\alpha) = C\sqrt{\alpha}$, then $H_2(v) = -y\frac{C^2 v^2}{4}$ and (19) becomes

$$\begin{aligned}0 = &(r-b)V + (b+k)yV' - \frac{y}{4}C^2(V')^2 - \frac{\sigma^2 y^2}{2}V'' \\ &+ \frac{ky}{\epsilon}\left(\epsilon c - 1 + \frac{ky}{2}\right) - \frac{(\epsilon(V'+c)-1+ky)_+^2}{2\epsilon} - \frac{\sigma^2 y^2}{2}V'' = 0.\end{aligned} \tag{20}$$

The demand law is of the form

$$D(X, p) = Xp^{-s}. \tag{21}$$

In this case, (13) becomes

$$\begin{aligned} 0 =& - rv + k1_{\{v+c\le (ky)^{-1/s}\}} \left((ky)^{-1/s} - c - v - yv'\right) - 1_{\{v+c>(ky)^{-1/s}\}}(v+c)^{-s}v' \\ &+ \frac{d}{dy}\left(y \max_{\alpha}(v\phi(\alpha) - \alpha)\right) + (\sigma^2 - b)yv' + \frac{\sigma^2 y^2}{2} v''. \end{aligned} \tag{22}$$

and the Hamiltonian H_1 takes the form:

$$H_1(y, v) = k1_{\{v+c\le(ky)^{-1/s}\}} \left(y(v+c) - \frac{s}{s-1}k^{-1/s}y^{1-1/s}\right) + \frac{1}{1-s}1_{\{v+c>(ky)^{-1/s}\}}(c+v)^{1-s}.$$

If $\phi(\alpha) = C\sqrt{\alpha}$, then the Hamilton Jacobi equation is:

$$\begin{aligned} 0 =&(r-b)V + byV' - \frac{y}{4}C^2(V')^2 - \frac{\sigma^2 y^2}{2}V'' \\ &+ k1_{\{V'+c\le(ky)^{-1/s}\}}\left(y(V'+c) - \frac{s}{s-1}k^{-1/s}y^{1-1/s}\right) + \frac{1}{1-s}1_{\{V'+c>(ky)^{-1/s}\}}(c+V')^{1-s}. \end{aligned} \tag{23}$$

2.2 A model of an open industry with a single technology

Here, investment into prospection is not reserved to the existing units. The efficiency of the investment is assumed to have the same law has above: an investment rate of α results in increasing the reserves at a rate $\phi(\alpha)$. The global increase of the reserves if all units invest α (the global investment is then αR) during dt is $R\phi(\alpha)dt$. Globally, since there are no restriction to investment, the investment rate will be such that the marginal value created by a unitary investment is 1, i.e.

$$u\phi'(\alpha^*) = 1, \tag{24}$$

where u is the value of a production unit. If we assume that $\phi(\alpha) = C\sqrt{\alpha}$, (24) has a unique solution: $\alpha^* = C^2u^2/4$. The evolution of the reserve is given by (10) with the same value of α^*. All the other aspects of the model are described in § 2.1. The value u is obtained by optimizing on the control β:

$$u(R, X) = (1 - rdt) \max_{0\le\beta\le k} \mathbb{E}\left(\ \beta g(R, X, u)dt + (1 - \beta dt)u(R + dR, X + dX)\ \right)$$

A first order expansion yields

$$-ru(R, X) + k(g(R, X, u) - u(R, X)) + \partial_R u \frac{dR}{dt} + \left(bX\partial_X u + \frac{1}{2}\sigma^2 X^2 \partial_{XX} u\right) = 0$$

where the optimal β has been given by $\beta^* = k1_{\{g(R,X,u)-u>0\}}$. We obtain the partial differential equation

$$-ru + k\left(g(\cdot, u) - u\right) - Q^*(\cdot, u)\partial_R u + \phi(\alpha^*)\partial_R u + bX\partial_X u + \frac{1}{2}\sigma^2 X^2 \partial_{XX} u = 0. \tag{25}$$

Looking for a nonnegative solution of (11) of the form $u(R, X) = v(y)$, we obtain the second order differential equation:

$$
\begin{aligned}
0 = & - rv + k1_{\{\tilde{D}^{-1}(ky) \geq v+c\}} \left(\tilde{D}^{-1}(ky) - c - v - yv' \right) - 1_{\{\tilde{D}(v+c)<ky\}} \tilde{D}(v + c)v' \\
& + y\phi(\alpha^*)v' + (\sigma^2 - b)yv' + \frac{\sigma^2 y^2}{2} v'',
\end{aligned}
\tag{26}
$$

and $\phi(\alpha^*) = C^2 v/2$ if $\phi(\alpha) = C\sqrt{\alpha}$.
In this case, there is no connected Hamilton-Jacobi equation.

3 Mathematical analysis of Hamilton-Jacobi equations related to (23)

The typical equation that we shall study is a simplified version of (23), namely

$$
v + H\left(x, \frac{dv}{dx}\right) = x^{-\alpha} \quad \text{in } [0, +\infty), \tag{27}
$$

where α is a nonnegative number.

Remark 3.1. *Note that v plays the role of $-V$ in (23). Compared to (23), we dropped the second order term and isolated the singularity of the data at $x = 0$. In fact, we will tackle (23) in § 3.3.*

Here, the Hamiltonian H is a continuous real valued function on $[0, +\infty) \times \mathbb{R}$ and satisfies the following assumptions: for some $m \geq 1$ and $\delta > 0$,

($\mathbf{H_1}$) there exist two positive constants $\nu \leq \mu$ and a nonnegative function $C \in \mathrm{BC}([0, +\infty))$ with $C(0) = 0$ such that

$$
\nu x|p|^m - C(x) \leq H(x, p) \leq \mu x|p|^m + C(x), \qquad \forall x \in [0, +\infty), \forall p \in \mathbb{R} \tag{28}
$$

($\mathbf{H_2}$) there exists a modulus of continuity ω such that

$$
|H(x, p) - H(y, p)| \leq \omega(|x - y|(1 + |p|^m)), \qquad \forall x, y \in [0, +\infty), \forall p \in \mathbb{R} \tag{29}
$$

If $m > 1$, there exists a modulus of continuity ω such that $\forall x, y \in [0, +\infty)$, $\forall p \in \mathbb{R}$,

$$
\left| H\left(x, px^{-\frac{1}{m}}\right) - H\left(y, py^{-\frac{1}{m}}\right) \right| \leq \omega\left(\left| x^{\frac{m-1}{m}} - y^{\frac{m-1}{m}} \right| (1 + |p|) \right) \tag{30}
$$

($\mathbf{H_3}$) For any $R > 0$, there exists a modulus of continuity ω_R such that $\forall x \in [0, +\infty)$, for any $p, q \in \mathbb{R}$, if $-R \leq p \leq R$ and $-R \leq q \leq R$, then

$$
|H(x, p) - H(x, q)| \leq \omega_R(x|p - q|) \tag{31}
$$

($\mathbf{H_4}$) if $m > 1$, there exist two continuous Hamiltonians H_1 and H_2 such that $H(x, p) = H_1(x, p) + H_2(x, p)$ in $[0, \delta] \times \mathbb{R}$, $x \mapsto H_1(x, p)$ is continuous in $[0, \delta]$ uniformly with respect to $p \in \mathbb{R}$, $x \mapsto H_2(x, p)$ is nondecreasing in $[0, \delta]$ for any $p \in \mathbb{R}$

An issue is to understand what definition of viscosity solutions should be chosen for (27), in particular what condition should be imposed at $x = 0$.

3.1 Bounded right hand sides

We first study

$$v + H\left(x, \frac{dv}{dx}\right) = f(x) \quad \text{in } [0, +\infty). \tag{32}$$

for $f \in \mathrm{BC}([0, +\infty))$.

3.1.1 The case when $m = 1$

In this paragraph, we suppose that Assumptions ($\mathbf{H_1}$), ($\mathbf{H_2}$), ($\mathbf{H_3}$) hold with $m = 1$. The following comparison result for (32) can be proved with the same techniques as in Lions [10], Bardi-Capuzzo Dolcetta [2]:

Theorem 3.1. *Consider two functions $u, w \in \mathrm{BUC}([0, +\infty))$. Under Assumptions ($\mathbf{H_1}$), ($\mathbf{H_2}$), ($\mathbf{H_3}$) with $m = 1$, if w is a supersolution of (32) and u is a subsolution of (32) in $[0, \infty)$, then $u \le w$.*

Proof. We double the variables in $[0, \infty]$ and use a similar penalty function as in the proof of [2], II, Theorem 3.5, namely $\frac{1}{\epsilon}|x - y|^2 + \beta(g(x) + g(y))$, where $g(x) = \frac{1}{2}\ln(1 + (x - a)_+^2)$ for some fixed value of $a > 0$ and $0 < \beta < 1$. The parameter β will be chosen later. We study the maximum points of $\psi_\epsilon(x, y) = u(x) - w(y) - \frac{1}{\epsilon}|x - y|^2 - \beta(g(x) + g(y))$.
Assume by contradiction that $M = \sup(u - w) > 0$. Then there exists $x_0 > 0$ such that $b = u(x_0) - w(x_0) > 0$. We can always choose $\bar\beta < 1$ small enough such that for all $0 < \beta \le \bar\beta$, $\beta g(x_0) < b/4$. Let (x_ϵ, y_ϵ) be such that $\psi_\epsilon(x_\epsilon, y_\epsilon) = \max_{x,y} \psi_\epsilon(x, y) > b/2$. Clearly, $\frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^2 + \beta(g(x_\epsilon) + g(y_\epsilon))$ is bounded uniformly with respect to ϵ and $\beta \in (0, \bar\beta]$. Therefore, as ϵ tend to 0, we may assume that both x_ϵ and y_ϵ converge to some point $\bar x \in [0, +\infty)$. Moreover, $\lim_{\epsilon \to 0} \frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^2 = 0$ since $\psi_\epsilon(x_\epsilon, x_\epsilon) + \psi_\epsilon(y_\epsilon, y_\epsilon) \le 2\psi_\epsilon(x_\epsilon, y_\epsilon)$.
Let us first focus on the case when $\bar x \ge a$. For ϵ small enough, we may assume that $x_\epsilon > a/2$ and that $y_\epsilon > a/2$. We use the notations $q_1 = \frac{2}{\epsilon}(x_\epsilon - y_\epsilon) + \beta g'(x_\epsilon)$ and $q_2 = \frac{2}{\epsilon}(x_\epsilon - y_\epsilon) - \beta g'(y_\epsilon)$. Since u is a viscosity subsolution, $|q_1| \le C$, where C is the Lipschitz constant of u in $[a/2, +\infty)$ which is independent of ϵ and β. This implies that $2\frac{|x_\epsilon - y_\epsilon|}{\epsilon} \le C + 1$ since $\|g'\|_\infty \le 1$. Then $|q_2| \le C + 2$. Set $Q = C + 2$.
The viscosity inequalities are $u(x_\epsilon) + H(x_\epsilon, q_1) \le f(x_\epsilon)$ and $w(y_\epsilon) + H(y_\epsilon, q_2) \ge f(y_\epsilon)$. From Assumption ($\mathbf{H_3}$),

$$|H(x_\epsilon, q_1) - H(x_\epsilon, q_2)| \le \omega_Q(\beta x_\epsilon(|g'(x_\epsilon)| + |g'(y_\epsilon)|)). \tag{33}$$

Note that $x_\epsilon|g'(x_\epsilon)| \le 1 + a$ and that $x_\epsilon|g'(y_\epsilon)| \le 1 + a + |x_\epsilon - y_\epsilon| \le 2 + a$ if ϵ is small enough. Hence, $\beta x_\epsilon(|g'(x_\epsilon)| + |g'(y_\epsilon)| \le \beta(3 + 2a)$. From Assumption ($\mathbf{H_2}$),

$$|H(x_\epsilon, q_2) - H(y_\epsilon, q_2)| \le \omega\left(|x_\epsilon - y_\epsilon|(1 + Q)\right). \tag{34}$$

Subtracting the two viscosity inequalities, using (33) and (34) and letting ϵ tend to 0, using also the continuity of f, yields that $b/2 \le u(\bar x) - w(\bar x) \le \omega_Q(\beta(3 + 2a))$. For β small enough, this is a contradiction.
Now, we consider the case when $\bar x < a$. Then $u(\bar x) - w(\bar x) = M$. For ϵ small enough, we may assume that $x_\epsilon < a$ and $y_\epsilon < a$. For $q_\epsilon = \frac{x_\epsilon - y_\epsilon}{\epsilon}$, the viscosity inequalities are $u(x_\epsilon) + H(x_\epsilon, q_\epsilon) \le f(x_\epsilon)$ and $w(y_\epsilon) + H(y_\epsilon, q_\epsilon) \ge f(y_\epsilon)$. Subtracting and using Assumption ($\mathbf{H_2}$) yields $u(x_\epsilon) - w(y_\epsilon) \le f(x_\epsilon) - f(y_\epsilon) + \omega(|x_\epsilon - y_\epsilon|(1 + |q_\epsilon|))$. Since $\lim_{\epsilon \to 0} |x_\epsilon - y_\epsilon||q_\epsilon| = 0$, we obtain that $M \le 0$ by passing to the limit. $\square$

An existence result can also be obtained. We skip the proof for brevity.

Theorem 3.2. *Under Assumptions (**H₁**), (**H₂**), (**H₃**) with $m = 1$, there exists a unique viscosity solution $v \in \mathrm{BUC}([0, +\infty))$ of (32).*

Value at $x = 0$ Recall that from the assumptions, $H(x,p) \le \mu x|p| + C(x)$ for a continuous function C vanishing at 0. Consider the following optimal control problem:

$$u(x) = \inf \left\{ \int_0^{+\infty} e^{-s} \left(f(z_x(s;\alpha)) - C(z_x(s;\alpha)) \right) ds; \quad \left| \begin{array}{l} \dot{z}_x(s;\alpha) = \alpha(s) z_x(s;\alpha) \\ z_x(0;\alpha) = x \\ \alpha(s) = \pm\mu \end{array} \right. \right\}.$$

Clearly, $u(0) = f(0)$ and u is a viscosity subsolution of (32) with $H(x,p)$ replaced by $\mu x|p| + C(x)$. On the other hand, there exists a function $w \in \mathrm{BUC}([0,+\infty))$ such that $w(0) = f(0)$, $w(x) \ge -H(x,p) + f(x)$ for any $x \in [0,+\infty)$ and $p \in \mathbb{R}$. The function w is a viscosity supersolution of (32) in $[0,+\infty)$. Then, from Theorem 3.1, the bounded viscosity solution v of (32) is such that $u \le v \le w$: therefore $v(0) = f(0)$.

3.1.2 The case when $m > 1$

Proposition 3.1. *Under Assumptions (**H₁**),(**H₂**), (**H₃**) with $m > 1$, there exists a viscosity solution $v \in \mathrm{BUC}([0,+\infty))$ of (32). It satisfies $v(0) \le f(0)$.*

Proof. For a positive number M, we approximate (32) by

$$v + H_M(x, \frac{dv}{dx}) = f(x), \quad \text{in } [0,+\infty) \tag{35}$$

where $H_M(x,p) = \min(\nu M x|p| + H(x,0), H(x,p))$.
Take $x, y \in [0,+\infty)$ and $p \in \mathbb{R}$: we can make out three cases:

1. If $H_M(x,p) = H(x,p)$ and $H_M(y,p) = H(y,p)$, then $\nu|p|^m \le \nu M|p| + H(x,0) + C(x) \le \nu M|p| + 2C(x) \le \nu M|p| + 2\|C\|_\infty$. Hence, $|H_M(x,p) - H_M(y,p)| \le \omega(|x-y|(1 + M|p| + 2\|C\|_\infty/\nu))$.
2. If $H_M(x,p) = \nu M x|p| + H(x,0)$ and $H_M(y,p) = \nu M y|p| + H(y,0)$, then $|H_M(x,p) - H_M(y,p)| \le \nu M|x-y||p| + \omega(|x-y|)$.
3. If $H_M(x,p) = \nu M x|p| + H(x,0)$ and $H_M(y,p) = H(y,p)$, then there must exist some z between x and y such that $H(z,p) = \nu M z|p| + H(z,0)$. In that case,

$$\begin{aligned} |H_M(x,p) - H_M(y,p)| &\le |H(y,p) - H(z,p)| + \nu M|x-z||p| + \omega(|x-z|) \\ &\le \omega(|z-y|(1 + M|p| + 2\|C\|_\infty/\nu)) + \nu M|x-z||p| + \omega(|x-z|) \\ &\le \omega(|x-y|(1 + M|p| + 2\|C\|_\infty/\nu)) + \nu M|x-y||p| + \omega(|x-y|). \end{aligned}$$

To summarize, H_M satisfies Assumption (**H₂**) with $m = 1$.
It is also clear that if $|p| \le R$ and $|q| \le R$, then for any $x \ge 0$, $|H_M(x,p) - H_M(x,q)| \le \nu M x|p-q| + \omega_R(x|p-q|)$ so Assumption (**H₃**) is satisfied by H_M.
Finally,

$$\nu x \min(M|p|, |p|^m) - C(x) \le H_M(x,p) \le \nu M x|p| + C(x), \quad \forall x \ge 0, \ \forall p \in \mathbb{R}. \tag{36}$$

This property is not exactly Assumption ($\mathbf{H_1}$) with $m = 1$, but it is sufficient for Theorems 3.1 and 3.2 to hold.

We therefore know that there is a unique viscosity solution $v_M \in \mathrm{BUC}([0, +\infty))$ of (35). Moreover $v_M(0) = f(0)$.

The comparison principle ensures that the family of functions $(v_M)_{M>0}$ is nonincreasing with respect to M: $v_N \le v_M$ if $N \ge M$, and that $v_N \ge \inf_x(f(x) - H(x, 0))$.

It can be deduced from the latter observation that for all $y > 0$, v_M is a viscosity subsolution of $\min(\nu|\frac{dv_M}{dx}|^m, \nu M|\frac{dv_M}{dx}|) \le c_y$ in $[y, +\infty)$, where c_y depends on y but not on M. If $M > (c_y/\nu)^{1-1/m}$, then v_M is a viscosity subsolution of $|v'_M| \le (c_y/\nu)^{1/m}$ in $[y, +\infty)$: therefore v_M is continuous in $[y, +\infty)$ uniformly with respect to M.

The sequence v_M converges in a monotone way to a function v defined on $(0, +\infty)$, and the convergence is uniform in the compact subsets of $(0, +\infty)$. Therefore $v \in \mathrm{BC}(0, \infty)$ and $v \ge \inf_x(f(x) - H(x, 0))$. By standard stability results, v is a viscosity solution of (35) in $(0, +\infty)$. For $c = \sup_x f(x) - \inf_x(f(x) - H(x, 0))$, v is a viscosity subsolution of $x|v'|^m \le \frac{c}{\nu}$ in $(0, +\infty)$, which implies that

$$|v(x) - v(y)| \le \left(\frac{c}{\nu}\right)^{\frac{1}{m}} \frac{m}{m-1} |x^{1-\frac{1}{m}} - y^{1-\frac{1}{m}}|, \qquad \forall 0 < x, y.$$

This shows that v is uniformly continuous on $(0, z]$ for all $z > 0$ and can be extended to a continuous function (still named v) defined in $[0, +\infty)$. Moreover, since $v_M(0) = f(0)$ and v_M converges in a nonincreasing manner in $(0, +\infty)$, $v(0) \le f(0)$. Hence, v is a subsolution of (32) in $[0, +\infty)$. We notice that $v(0) = \liminf_{x\to 0^+, M\to+\infty} v_M(x)$.

We now claim that one among the following two assertions is true

1. $v(0) = f(0)$
2. $v(0) < f(0)$ and for any function $\phi \in C^1([0, +\infty))$, $v - \phi$ does not have a local minimum at 0.

As a consequence, v is a supersolution of (32) in $[0, +\infty)$.

The claim is proved by contradiction: assume that $v(0) < f(0)$ and that $v - \phi$ has a local minimum at 0. Replacing possibly ϕ by $\phi - x^2$, we may also suppose that $v - \phi$ has a strict local minimum at 0. Since $v(0) = \liminf_{x\to 0^+, M\to+\infty} v_M(x)$, classical arguments show that there exists a sequence $(M_n)_{n>0}$ such that $\lim_{n\to\infty} M_n = +\infty$, a sequence of positive numbers $(x_n)_{n>0}$ such that $v_{M_n} - \phi$ has a local minimum at x_n, $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} v_{M_n}(x_n) = v(0)$. A key point is to observe that there cannot exist a subsequence (not relabeled) such that $x_n = 0$, because in this case, $v_{M_n}(0) = f(0)$ would imply $v(0) = f(0)$. Therefore, $v_{M_n}(x_n) + H_{M_n}\left(x_n, \frac{d\phi}{dx}(x_n)\right) \ge f(x_n)$, and since $\phi \in C^1([0, +\infty))$, this yields for n large enough: $v_{M_n}(x_n) + H\left(x_n, \frac{d\phi}{dx}(x_n)\right) \ge f(x_n)$. Letting $n \to \infty$ yields that $\lim_{n\to\infty} \left|\frac{d\phi}{dx}(x_n)\right| = +\infty$, i.e. the desired contradiction. □

Theorem 3.3. *Under Assumptions ($\mathbf{H_1}$), ($\mathbf{H_2}$), ($\mathbf{H_3}$) and ($\mathbf{H_4}$) with $m > 1$, if $w \in \mathrm{BUC}([0, +\infty))$ is a supersolution of (32) and $u \in \mathrm{BUC}([0, +\infty))$ is a subsolution of (32) in $[0, +\infty)$, then $u \le w$.*

Proof. In the spirit of the comparison principles proved by Soner [12, 13], Capuzzo Dolcetta-Lions [3], see also [2], for problems with state constraint boundary conditions, we choose a monotone function $T \in C^1([0, +\infty))$ such that $T(x) = 1$ in $[0, h]$, and $T(x) = 0$ in $[2h, +\infty)$, for some $h < \delta$.

Assume first that the supremum M of $u - w$ is not achieved at $x = 0$. In this case, the same proof as for Theorem 3.1 can be used, mainly because H is uniformly coercive away from $x=0$: it consists of studying the maximum points of the function $\psi_\epsilon(x,y) = u(x) - w(y) - \frac{1}{\epsilon}|x-y|^2 - \beta(g(x)+g(y))$, where $g(x) = \frac{1}{2}\ln(1+(x-\delta)_+^2)$. It yields that $M \le 0$.

Assume now that the maximum M of $u-w$ is achieved at $x=0$ and consider the function $\psi_\epsilon(x,y) = u(x) - w(y) - \frac{1}{\epsilon}\left|x^{\frac{m-1}{m}} - (y+\sqrt{\epsilon}T(y))^{\frac{m-1}{m}}\right|^2 - (g(x)+g(y))$. The supremum of ψ_ϵ in $[0,+\infty)^2$ is a maximum. Since $u(0)-w(0)=M$, we see that $\max \psi_\epsilon(x,y) \ge \psi_\epsilon(\sqrt{\epsilon},0) \ge M - \omega_u(\sqrt{\epsilon})$, where ω_u is the modulus of continuity of u. Therefore, if (x_ϵ, y_ϵ) a maximum point of ψ_ϵ, then x_ϵ, y_ϵ and $\frac{1}{\epsilon}\left|x^{\frac{m-1}{m}} - (y+\sqrt{\epsilon}T(y))^{\frac{m-1}{m}}\right|^2$ are bounded by some R independent of ϵ. Therefore, after the extraction of a subsequence, we can assume that both x_ϵ and y_ϵ converge to some point $\bar{x} \in [0,+\infty)$. Moreover,

$$\psi_\epsilon(x_\epsilon, y_\epsilon) \le M + \omega_w(|x_\epsilon - y_\epsilon|) - \frac{1}{\epsilon}\left|(y_\epsilon + \sqrt{\epsilon}T(y_\epsilon))^{\frac{m-1}{m}} - x_\epsilon^{\frac{m-1}{m}}\right|^2 - (g(x_\epsilon)+g(y_\epsilon)),$$

where ω_w is the modulus of continuity of w. Combining the latter two observations,

$$\frac{1}{\epsilon}\left|(y_\epsilon + \sqrt{\epsilon}T(y_\epsilon))^{\frac{m-1}{m}} - x_\epsilon^{\frac{m-1}{m}}\right|^2 + (g(x_\epsilon)+g(y_\epsilon)) \le \omega_w(|x_\epsilon - y_\epsilon|) + \omega_u(\sqrt{\epsilon}),$$

which implies that

$$\left|(y_\epsilon + \sqrt{\epsilon}T(y_\epsilon))^{\frac{m-1}{m}} - x_\epsilon^{\frac{m-1}{m}}\right| \le \epsilon^{\frac{1}{2}}\eta(\epsilon) \quad \text{with } \lim_{\epsilon\to 0}\eta(\epsilon) = 0, \tag{37}$$

and that

$$g(x_\epsilon) + g(y_\epsilon) \le \eta(\epsilon). \tag{38}$$

If for a subsequence, $y_\epsilon \in (h,R]$, then we may assume that $x_\epsilon \in (h/2,R]$. Of course, $\bar{x} \in [h,R]$. We then use the Lipschitz continuity of u in $[h/2,+\infty)$ and obtain that

$$\left|\frac{2}{\epsilon}\frac{m-1}{m}\left(x_\epsilon^{\frac{m-1}{m}} - (y_\epsilon + \sqrt{\epsilon}T(y_\epsilon))^{\frac{m-1}{m}}\right) x_\epsilon^{-\frac{1}{m}} + g'(x_\epsilon)\right| \le L_u,$$

where L_u is the Lipschitz constant of u in $[h/2,+\infty)$. Next, setting

$$\begin{aligned}
q &= \frac{2}{\epsilon}\frac{m-1}{m}\left(x_\epsilon^{\frac{m-1}{m}} - (y_\epsilon + \sqrt{\epsilon}T(y_\epsilon))^{\frac{m-1}{m}}\right) - g'(y_\epsilon)y_\epsilon^{\frac{1}{m}},\\
q_1 &= \frac{2}{\epsilon}\frac{m-1}{m}\left(x_\epsilon^{\frac{m-1}{m}} - (y_\epsilon + \sqrt{\epsilon}T(y_\epsilon))^{\frac{m-1}{m}}\right) x_\epsilon^{-\frac{1}{m}} + g'(x_\epsilon),\\
q_2 &= \frac{2}{\epsilon}\frac{m-1}{m}\left(x_\epsilon^{\frac{m-1}{m}} - (y_\epsilon + \sqrt{\epsilon}T(y_\epsilon))^{\frac{m-1}{m}}\right)(y_\epsilon + \sqrt{\epsilon}T(y_\epsilon))^{-\frac{1}{m}}(1+\sqrt{\epsilon}T'(y_\epsilon)) - g'(y_m),\\
\tilde{q}_1 &= \frac{2}{\epsilon}\frac{m-1}{m}\left(x_\epsilon^{\frac{m-1}{m}} - (y_\epsilon + \sqrt{\epsilon}T(y_\epsilon))^{\frac{m-1}{m}}\right) x_\epsilon^{-\frac{1}{m}} - g'(y_\epsilon)y_\epsilon^{\frac{1}{m}}x_\epsilon^{-\frac{1}{m}},\\
\tilde{q}_2 &= \frac{2}{\epsilon}\frac{m-1}{m}\left(x_\epsilon^{\frac{m-1}{m}} - (y_\epsilon + \sqrt{\epsilon}T(y_\epsilon))^{\frac{m-1}{m}}\right) y_\epsilon^{-\frac{1}{m}} - g'(y_\epsilon),
\end{aligned}$$

the previous estimates show that there exist positive constant Q and C independent of ϵ (if ϵ is small enough) such that $\max(|q|,|q_1|,|q_2|,|\tilde{q}_1|,|\tilde{q}_2|) \le Q$ and that $|q_2 - \tilde{q}_2| \le C\sqrt{\epsilon}$.

The viscosity inequalities are $u(x_\epsilon) + H(x_\epsilon, q_1) \le f(x_\epsilon)$ and $w(y_\epsilon) + H(y_\epsilon, q_2) \ge f(y_\epsilon)$. Then,

$$
\begin{aligned}
|H(x_\epsilon, q_1) - H(x_\epsilon, \tilde q_1)| &\le \omega_Q\left(|x_\epsilon|\left(g(x_\epsilon) + g(y_\epsilon) y_\epsilon^{\frac{1}{m}} x_\epsilon^{-\frac{1}{m}}\right)\right), \quad (39)\\
|H(y_\epsilon, q_2) - H(y_\epsilon, \tilde q_2)| &\le \omega_Q(C|y_\epsilon|\sqrt{\epsilon}), \quad (40)\\
|H(x_\epsilon, \tilde q_1) - H(y_\epsilon, \tilde q_2)| &\le \omega\left(\left|x_\epsilon^{\frac{m-1}{m}} - y_\epsilon^{\frac{m-1}{m}}\right|(1+|q|)\right). \quad (41)
\end{aligned}
$$

Indeed (39) and (40) come from Assumption $(\mathbf{H_3})$ and (41) is a consequence of Assumption $(\mathbf{H_2})$, namely (30). The right hand sides of (39)-(41) tend to 0 as $\epsilon \to 0$. Subtracting the viscosity inequalities, using (39)-(41) and (38), then by passing to the limit yield that

$$
M = u(\bar x) - w(\bar x) \le 0.
$$

There remains to discuss the case when for ϵ small enough, $y_\epsilon \in [0, h)$. In this case $T(y_\epsilon) = 1$, $T'(y_\epsilon) = 0$ and $g(y_\epsilon) = g(x_\epsilon) = 0$ for ϵ small enough. We make out two cases:

1. $y_\epsilon > 0$: defining $q_\epsilon = \frac{2}{\epsilon}\frac{m-1}{m}\left(x_\epsilon^{\frac{m-1}{m}} - (y_\epsilon + \sqrt{\epsilon})^{\frac{m-1}{m}}\right)$ and subtracting the viscosity inequalities, we obtain

$$
u(x_\epsilon) - w(y_\epsilon) + H\left(x_\epsilon, q_\epsilon x_\epsilon^{-\frac{1}{m}}\right) - H\left(y_\epsilon, q_\epsilon (y_\epsilon + \sqrt{\epsilon})^{-\frac{1}{m}}\right) \le f(x_\epsilon) - f(y_\epsilon). \quad (42)
$$

   Then

$$
\begin{aligned}
& H\left(x_\epsilon, q_\epsilon x_\epsilon^{-\frac{1}{m}}\right) - H\left(y_\epsilon, q_\epsilon (y_\epsilon + \sqrt{\epsilon})^{-\frac{1}{m}}\right)\\
\ge\ & -\left|H\left(x_\epsilon, q_\epsilon x_\epsilon^{-\frac{1}{m}}\right) - H\left(y_\epsilon + \sqrt{\epsilon}, q_\epsilon (y_\epsilon + \sqrt{\epsilon})^{-\frac{1}{m}}\right)\right|\\
& -\left|H_1\left(y_\epsilon + \sqrt{\epsilon}, q_\epsilon (y_\epsilon + \sqrt{\epsilon})^{-\frac{1}{m}}\right) - H_1\left(y_\epsilon, q_\epsilon (y_\epsilon + \sqrt{\epsilon})^{-\frac{1}{m}}\right)\right|\\
& + H_2\left(y_\epsilon + \sqrt{\epsilon}, q_\epsilon (y_\epsilon + \sqrt{\epsilon})^{-\frac{1}{m}}\right) - H_2\left(y_\epsilon, q_\epsilon (y_\epsilon + \sqrt{\epsilon})^{-\frac{1}{m}}\right).
\end{aligned} \quad (43)
$$

   The first term in the right hand side of (43) tends to 0 from (30) and (37). The second term tends to 0 using the continuity of H_1, see Assumption $(\mathbf{H_4})$. The third term is nonnegative since H_2 is nondecreasing w.r.t x by Assumption $(\mathbf{H_4})$.
   Combining these observations, we deduce from (42) that $M \le 0$ by letting ϵ tend to 0.

2. If $y_\epsilon = 0$, then $y \mapsto w(y) + \frac{1}{\epsilon}\left|x^{\frac{m-1}{m}} - (y + \sqrt{\epsilon})^{\frac{m-1}{m}}\right|^2$ has a minimum at 0: this implies that $w(0) \ge f(0)$ and that $M \le 0$ since $u(0) \le f(0)$.

□

3.2 Analysis of (27)

We still make Assumptions $(\mathbf{H_1})$, $(\mathbf{H_2})$, $(\mathbf{H_3})$ and $(\mathbf{H_4})$ with $m > 1$ and suppose furthermore that $0 < \alpha < m - 1$.

Definition 3.1. *We say that $v \in \mathrm{BUC}([0, +\infty))$ is a viscosity solution of (27) if it is a viscosity solution of (27) in $(0, +\infty)$ and if it is not possible to find a C^1 function ϕ such that $v - \phi$ has a local minimum at 0.*

Proposition 3.2. *Under Assumptions $(\mathbf{H_1})$, $(\mathbf{H_2})$, $(\mathbf{H_3})$ and $(\mathbf{H_4})$ with $m > 1$, if $0 < \alpha < m - 1$, there exists a viscosity solution $v \in \mathrm{BUC}([0, +\infty))$ of (27).*

Proof. There exist four constants k_1, $k_2 > 0$, $k_3 = -\sup_x H(x,0)$, $\bar{h}$: $0 < \bar{h} < \delta$ (where δ is the constant appearing in the assumptions), such that $u_h(x) = \max(k_1 - k_2(x+h)^{(m-1-\alpha)/m}, k_3)$ is a subsolution of (27) in $(0,+\infty)$, for all h, $0 < h < \bar{h}$. Calling $R(h) = k_1 + 2\mu(k_2(m-1-\alpha)/m)^m h^{-\alpha}$, it is always possible to decrease $\bar{h}$ in such a way that for any $0 < h < \bar{h}$, $R(h) > 0$ and u_h is also a subsolution of $u_h + H\left(x, \frac{du_h}{dx}\right) \le \min(x^{-\alpha}, R(h))$ in $[0,+\infty)$. From the previous paragraph, we know that there exists a unique viscosity solution $v_h \in \mathrm{BUC}([0,+\infty))$ of

$$v_h + H\left(x, \frac{dv_h}{dx}\right) = \min(x^{-\alpha}, R(h)) \quad \text{in } [0,+\infty). \tag{44}$$

Comparison results imply that $v_h \ge k_3$ and that the family $(v_h)_h$ is nonincreasing with respect to h.
It is also possible to find a bounded supersolution w of (27) in $(0,\infty)$ of the form $w(x) = K_1 - K_2\min(x,1)^{(m-1-\alpha)/m}$. Note that w is also a supersolution of (44) in $[0,+\infty)$.
Hence, $k_3 \le u_h \le v_h \le w$, and we see that $|v_h|$ is bounded uniformly in h; furthermore, v_h is a viscosity subsolution of $\nu x\left|\frac{dv_h}{dx}\right|^m \le x^{-\alpha} - k_3 + C(x)$; this shows that for any $y > 0$, the norms $\|v_h\|_{C^{(m-1-\alpha)/m}([0,y])}$ are bounded uniformly with respect to h.
From the monotonicity of the sequence $(v_h)_h$ and the uniform Hölder estimate, v_h converges uniformly in the intervals $[0,y]$, $y > 0$ to some $\underline{v} \in \mathrm{BC}([0,\infty))$ such that $\underline{v} \in C^{(m-1-\alpha)/m}([0,y])$ for any $y > 0$. The function $\underline{v}$ is a viscosity solution of (27) in $(0,+\infty)$ and $\underline{v} \in \mathrm{BUC}([0,\infty))$. We also claim that it is not possible to find a C^1 function ϕ such that $\underline{v} - \phi$ has a local minimum at 0. Indeed, in the opposite case, we could always assume that the minimum is strict by replacing ϕ by $\phi - x^2$, and by standard arguments, we could find a sequence $(h_n)_{n>0}$ such that $\lim_{n\to\infty} h_n = 0$, a sequence of positive numbers $(x_n)_{n>0}$ such that $v_{h_n} - \phi$ has a local minimum at x_n, $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} v_{h_n}(x_n) = \underline{v}(0)$. This would imply that $v_{h_n}(x_n) + H\left(x_n, \frac{d\phi}{dx}(x_n)\right) \ge \min(x_n^{-\alpha}, R(h_n))$. This would yield that $\lim_{n\to\infty}\left|\frac{d\phi}{dx}(x_n)\right| = +\infty$, the desired contradiction. We have proved that $\underline{v}$ is a solution of (27).
Moreover, if $\tilde{v} \in \mathrm{BUC}([0,+\infty))$ is another viscosity solution of (27), then $\tilde{v}$ is a supersolution of (44). Hence $\tilde{v} \ge v_h$, which shows that $\tilde{v} \ge \underline{v}$: $\underline{v}$ is the minimal solution of (27). □

Proposition 3.3. *Under Assumptions* **(H₁)**, **(H₂)**, **(H₃)** *and* **(H₄)** *with* $m > 1$, *if* $w \in \mathrm{BUC}([0,+\infty))$ *is a supersolution of (27) and* $u \in \mathrm{BUC}([0,+\infty))$ *is a subsolution of (27), then* $u \le w$.

Proof. The proof is identical to that of Theorem 3.3 until the discussion of the case when $y_\epsilon \in [0,h]$; at this point, the proof slightly differs as follows:

1. if $y_\epsilon > 0$, then, subtracting the viscosity inequalities, we get

$$u(x_\epsilon) - w(y_\epsilon) + H\left(x_\epsilon, q_\epsilon x_\epsilon^{-\frac{1}{m}}\right) - H\left(y_\epsilon, q_\epsilon(y_\epsilon + \sqrt{\epsilon})^{-\frac{1}{m}}\right) \le x_\epsilon^{-\alpha} - y_\epsilon^{-\alpha}, \tag{45}$$

which is the counterpart of (42), recalling that $q_\epsilon = \frac{2}{\epsilon}\frac{m-1}{m}\left(x_\epsilon^{\frac{m-1}{m}} - (y_\epsilon + \sqrt{\epsilon})^{\frac{m-1}{m}}\right)$.
But we also know that

$$\left|(y_\epsilon + \sqrt{\epsilon})^{\frac{m-1}{m}} - x_\epsilon^{\frac{m-1}{m}}\right| \le \epsilon^{\frac{1}{2}}\eta(\epsilon) \quad \text{with } \lim_{\epsilon\to 0}\eta(\epsilon) = 0,$$

which implies that $x_\epsilon > y_\epsilon$ and that $x_\epsilon^{-\alpha} \le y_\epsilon^{-\alpha}$ for ϵ small enough. From this, the fact that $M \le 0$ follows as in the proof of Theorem 3.3.

2. The case $y_\epsilon = 0$ is not possible since otherwise $y \mapsto w(y) + \frac{1}{\epsilon}\left|x^{\frac{m-1}{m}} - (y+\sqrt{\epsilon})^{\frac{m-1}{m}}\right|^2$ would have a minimum at 0.

□

Remark 3.2. *Under Assumptions ($\mathbf{H_1}$), ($\mathbf{H_2}$), ($\mathbf{H_3}$) and ($\mathbf{H_4}$) with $m > 1$, if $\alpha \geq m-1$, it would be possible to prove that there exists a unique viscosity solution $v \in C(0,+\infty)$ of (27) which blows up at 0 (like $x^{(m-1-\alpha)/m}$ if $\alpha > m-1$ and logarithmically if $\alpha = m-1$). For brevity, we do no study this case in details; the proof of the existence of a minimal viscosity solution would be rather similar to the proof of Proposition 3.2. For the proof of uniqueness, a different strategy close to the one introduced in [6] would be needed.*

3.3 Analysis of (23) in the case when $\sigma = 0$

We aim at proving existence and uniqueness of a nondecreasing solution of (23), focusing on the case $\sigma = 0$ for simplicity. After the change of variables $v = -V$, $x = y$, the equation takes the form

$$\alpha v + H_1(x, v') + H_2(x, v') = \frac{s}{1-s} k^{1-1/s} x^{1-1/s}, \tag{46}$$

where $H_1(x,p) = \beta x p + \gamma x p^2$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $s < 1$ and

$$H_2(x,p) = 1_{\{c-p>(kx)^{-1/s}\}}\left(-\frac{1}{1-s}(c-p)^{1-s} + \frac{s}{1-s}k^{1-1/s}x^{1-1/s}\right) + 1_{\{c-p\leq(kx)^{-1/s}\}} kx(p-c),$$

with $k > 0$, $c > 0$.

Remark 3.3. *Note that $H_2 \in C^1([0,+\infty)\times\mathbb{R})$ and that for any $p \in \mathbb{R}$, $x \mapsto H_2(x,p) + \frac{s}{s-1}k^{1-1/s}x^{1-1/s}$ is nondecreasing with respect to x.*

Theorem 3.4. *There exists a unique nonincreasing viscosity solution $v \in \mathrm{BUC}([0,+\infty))$ of (46).*

Proof. Since we look for a nonincreasing function v, we first modify the equation as follows:

$$0 = \alpha v + \widetilde{H}(x, v') = \frac{s}{1-s}k^{1-1/s}x^{1-1/s}, \tag{47}$$

where $\widetilde{H}(x,p) = H_1(x,p) + \widetilde{H}_2(x,p)$ and $\widetilde{H}_2(x,p) = H_2\left(x, p1_{\{p\leq 0\}}\right)$. From Remark 3.3, $\widetilde{H}_2 \in C([0,+\infty)\times\mathbb{R})$ and for any $p \in \mathbb{R}$, $x \mapsto \widetilde{H}_2(x,p) + \frac{s}{s-1}k^{1-1/s}x^{1-1/s}$ is nondecreasing with respect to x.
One can check that if $v \in \mathrm{BUC}([0,+\infty))$ is a nonincreasing viscosity solution of (46), then it is a viscosity solution of (47).
Existence and uniqueness for (47) will stem from Propositions 3.2 and 3.3, once we have checked that $\widetilde{H}$ satisfies Assumptions ($\mathbf{H_1}$)-($\mathbf{H_4}$) with $m = 2$.

- If $p > 0$, then $\widetilde{H}_2(x,p) = H_2(x,0)$ is a function in $\mathrm{BUC}([0,+\infty))$ that vanishes at 0. If $p \leq 0$ and $c - p > (kx)^{-1/s}$, then $-\frac{k}{1-s}x(c-p) \leq -\frac{1}{1-s}(c-p)^{1-s} \leq \widetilde{H}_2(x,p) \leq \frac{s}{1-s}k^{1-1/s}x^{1-1/s} \leq \frac{ks}{1-s}x(c-p)$. If $p \leq 0$ and $c - p \leq (kx)^{-1/s}$, then $kx \leq c^{-s}$, and $\widetilde{H}_2(x,p) = kx(p-c)$. Combining the preceding observations, we deduce that there exists a nonnegative function $\eta \in \mathrm{BUC}([0,+\infty))$ with $\eta(0) = 0$ and $\zeta > 0$, such that

$$-\eta(x) - \zeta x|p| \leq \widetilde{H}_2(x,p) \leq \eta(x) + \zeta x|p|. \tag{48}$$

  This implies that $\widetilde{H}$ satisfies ($\mathbf{H_1}$) with $m = 2$.

- We claim that $\widetilde{H}_2$ satisfies Assumption ($\mathbf{H_2}$) with $m = 1$. Consider $p \le 0$: if $c - p > (kx)^{-1/s}$ and $c - p > (ky)^{-1/s}$ then $|\widetilde{H}_2(x,p) - \widetilde{H}_2(y,p)| \le k \max((kx)^{-1/s}, (ky)^{-1/s})|x - y| \le k(c-p)|x-y|$. If $c - p \le (kx)^{-1/s}$ and $c - p \le (ky)^{-1/s}$ then $|\widetilde{H}_2(x,p) - \widetilde{H}_2(y,p)| = k|x-y||c-p|$. Finally, if $c - p \le (kx)^{-1/s}$ and $c - p > (ky)^{-1/s}$, there exists z between x and y such that $c - p = (kz)^{-1/s}$, and $|\widetilde{H}_2(x,p) - \widetilde{H}_2(y,p)| \le |\widetilde{H}_2(x,p) - \widetilde{H}_2(z,p)| + |\widetilde{H}_2(z,p) - \widetilde{H}_2(y,p)| \le k|x-y||c-p|$. Finally, if $p > 0$, then $|\widetilde{H}_2(x,p) - \widetilde{H}_2(y,p)| = |\widetilde{H}_2(x,0) - \widetilde{H}_2(y,0)| \le kc|x-y|$. The claim is proved. This implies that $\widetilde{H}$ satisfies Assumption ($\mathbf{H_2}$) with $m = 2$.

- We claim that $\widetilde{H}_2$ satisfies Assumption ($\mathbf{H_3}$) with $m = 1$. Set $\tilde{p} = p 1_{\{p \le 0\}}$ and $\tilde{q} = q 1_{\{q \le 0\}}$; if $c - \tilde{p} > (kx)^{-1/s}$ and $c - \tilde{q} > (kx)^{-1/s}$ $|\widetilde{H}_2(x,p) - \widetilde{H}_2(x,q)| = \frac{1}{1-s}|(c-\tilde{p})^{1-s} - (c-\tilde{q})^{1-s}| \le \max\left((c-\tilde{p})^{-s}, (c-\tilde{q})^{-s}\right)|\tilde{p} - \tilde{q}| \le kx|\tilde{p} - \tilde{q}| \le kx|p-q|$. If $c - \tilde{p} \le (kx)^{-1/s}$ and $c - \tilde{q} \le (kx)^{-1/s}$ then $|\widetilde{H}(x,p) - \widetilde{H}(x,q)| = kx|\tilde{p} - \tilde{q}| \le kx|p-q|$. Finally, if $c - \tilde{p} \le (kx)^{-1/s}$ and $c - \tilde{q} > (kx)^{-1/s}$, there exists r between $\tilde{p}$ and $\tilde{q}$ such that $c - r = (kx)^{-1/s}$, and this yields that $|\widetilde{H}(x,p) - \widetilde{H}(x,q)| \le kx|\tilde{p} - \tilde{q}| \le kx|p-q|$. The claim is proved. It implies that $\widetilde{H}$ satisfies Assumption ($\mathbf{H_3}$) with $m = 2$.

- There remains to study ($\mathbf{H_4}$). From (48), we see that there exists $\bar{p} > 0$ and $C > 0$ such that $\widetilde{H}(x,p) + Cx$ is nondecreasing with respect to x in $[0,\delta] \times \{|p| \ge \bar{p}\}$, because the dominating behavior is that of $\beta x |p|^m$ in this region. Let χ be a smooth monotone function on $\mathbb{R}_+$ such that $\chi(t) = 1$ for $t \ge 2\bar{p}$ and $\chi(t) = 0$ for $t \le \bar{p}$. We can split $\widetilde{H}$ as follows: $\widetilde{H}(x,p) = \widetilde{H}_3(x,p) + \widetilde{H}_4(x,p)$ where $\widetilde{H}_3(x,p) = \chi(|p|)\widetilde{H}(x,p) + Cx$ and $\widetilde{H}_4(x,p) = (1 - \chi(|p|))\widetilde{H}(x,p) - Cx$, which proves that $\widetilde{H}$ satisfies ($\mathbf{H_4}$).

Combining all these observations, we see that (47) has a unique solution $v \in \mathrm{BUC}([0,+\infty))$. This implies that there is at most one nonincreasing viscosity solution $v \in \mathrm{BUC}([0,+\infty))$ of (46).
We now need to prove that the function v that we just found is indeed nonincreasing: note first that v is Lipschitz continuous in $[\underline{x}, +\infty)$ for all $\underline{x} > 0$, and that the Lipschitz constant of v in $[\underline{x}, +\infty)$ tends to 0 as $\underline{x} \to +\infty$. This implies that for all $h > 0$, $\lim_{x\to\infty} |v(x+h) - v(x)| = 0$. Assume by contradiction that $\inf_{x \in [0,+\infty)} v(x) - v(x+h) = m < 0$. From the latter point, there exists $x_0 \ge 0$ such that $v(x_0) - v(x_0 + h) = m$. Consider the function $\phi(x,y) = v(x) - v(y) + \frac{|x+h-y|^2}{\epsilon}$. This function has a negative infimum (not greater than m). By studying maximizing sequences for ϕ and using the boundedness of v, it can be proved that the infimum is achieved by some pair (x_ϵ, y_ϵ). Standard arguments show that $\lim_{\epsilon\to 0} |x_\epsilon + h - y_\epsilon| = 0$ and that there exits $R > 0$ such that $x_\epsilon \in [0,R]$ for ϵ small enough. Therefore, we can extract a subsequence (not relabeled) such that $\lim_{\epsilon \to 0} x_\epsilon = \bar{x} \in [0,R]$ and $\lim_{\epsilon\to 0} y_\epsilon = \bar{x} + h$. Then, since $\phi(x_\epsilon, y_\epsilon) \le \phi(x_\epsilon, x_\epsilon + h)$, we see that $\lim_{\epsilon \to 0} \frac{|x_\epsilon + h - y_\epsilon|^2}{\epsilon} = 0$. From the viscosity inequalities, we deduce that

$$v(x_\epsilon) - v(y_\epsilon) + \widetilde{H}(x_\epsilon, 2\frac{y_\epsilon - h + x_\epsilon}{\epsilon}) + \widetilde{H}(y_\epsilon, 2\frac{y_\epsilon - h + x_\epsilon}{\epsilon}) + \frac{s}{s-1} k^{1-1/s}\left(x_\epsilon^{1-1/s} - y_\epsilon^{1-1/s}\right) \ge 0.$$

But $x \mapsto \widetilde{H}(x,p) + \frac{s}{s-1}k^{1-1/s}x^{1-1/s}$ is nondecreasing with respect to x, and that $y_\epsilon - x_\epsilon \ge h/2$ for ϵ small enough. Therefore, for ϵ small enough, $v(x_\epsilon) - v(y_\epsilon) \ge 0$, which contradicts the fact that $m < 0$.
We have thus proved that the viscosity solution v of (47) is nondecreasing. One can then check that v is the nondecreasing BUC viscosity solution of (46). $\square$

4 Extension: a closed industry with two technologies

Suppose now that there are two types of production units (different technologies), with different prospection and production costs. The indices $i = 1, 2$ will be used to distinguish the two kinds of production units.

- The total reserve of type i will be noted $R_i(t) \in \mathbb{R}_+$. With the assumption above, $R_i(t)$ can also be viewed as the quantity of production units of type i.
- It is assumed that the production capacity is proportional to the reserve. Let k, $0 < k < 1$ be the production capacity of a single production unit (with reserve 1): to begin with, k is assumed to be independent of the reserve type $i = 1, 2$.
- Let $c_i > 0$ be the unitary production cost (i.e. the production cost of a unit of ore) of the industry of type i, with $c_1 < c_2$.
- Each production unit of type i can invest into prospection. For the industry of type i, the flux invested into prospection by a single unit is $\alpha_i dt$ (α_i is a control parameter). An investment rate of α_i increases the reserves of type j, $j = 1, 2$ with a rate of $\phi_{i,j}(\alpha_i)$, where $\phi_{i,j}$ are increasing and concave functions on $\mathbb{R}_+$ with $\phi_{i,j}(0) = 0$. To begin with, it is possible to assume that $\phi_{i,j} = 0$ is $i \neq j$, i.e. the new reserves created by the industry of type i are only of type i. For example, one may choose $\phi_i(\alpha) = \sqrt{\alpha}$
- The other control parameters are the production rates β_i of a unit of type i, $i = 1, 2$, with $0 \leq \beta_i \leq k$.
- The discount factor of the expected income is $r_i > 0$ for the industry of type i.

4.1 The strategy of the production units

Let $u_i(R_1, R_2, X)$ be the expected benefit of a production unit of type i, discounted by r_i. As above, when a production unit of type i produces q units of ore, its production cost is qc_i and the cost of decreasing the reserves is $qu_i(R_1, R_2, X)$. The total cost is $qc_i + qu_i(R_1, R_2, X)$ and the income is pq. Therefore, the following inequality should hold: $p \geq c_i + u_i(R_1, R_2, X)$. If $p = c_i + u_i(R_1, R_2, X)$, it is indifferent for a unit of type i to produce or not.

Fixing the price p and the global productions Q_i, $i = 1, 2$ knowing $u_i(R_1, R_2, X)$ As above, the price p and global productions Q_i can be found by matching offer and demand. Let $P^*(R_1, R_2, X)$ be the price function. The cash income for a unit of ore produced by the industry of type i is $g_i(R_1, R_2, X, u_1, u_2) = P^*(R_1, R_2, X, u_1, u_2) - c_i$.
In order to divide the number of cases by two, we assume that $u_2 + c_2 > u_1 + c_1$, but should the opposite case occur, we would compute the prices and productions in a symmetric way, by exchanging the indices 1 and 2. If $u_2 + c_2 > u_1 + c_1$, there are four different cases:

1. **The two industries produce at full capacity** when $p > u_2 + c_2$.
   The total productions are $Q_1 = kR_1$ and $Q_2 = kR_2$. Matching offer and demand yields $k(R_1 + R_2) = X\tilde{D}(p)$, i.e $P^*(R_1, R_2, X, u_1, u_2) = \tilde{D}^{-1}\left(k\left(\frac{R_1}{X} + \frac{R_2}{X}\right)\right)$. The inequality is then equivalent to $\tilde{D}^{-1}\left(k\left(\frac{R_1}{X} + \frac{R_2}{X}\right)\right) > \max(u_1 + c_1, u_2 + c_2)$. In this regime, $g_i(R_1, R_2, X, u_1, u_2) = \tilde{D}^{-1}\left(k\left(\frac{R_1}{X} + \frac{R_2}{X}\right)\right) - c_i$.

2. **Industry 1 produces at full capacity and industry 2 has a positive but partial production** when $u_1 + c_1 < p = u_2 + c_2$. The total productions are $Q_1 = kR_1$ and Q_2 is obtained by matching offer and demand: $Q_2/X = \tilde{D}(c_2 + u_2) - kR_1/X$. Then $kR_2 > Q_2 > 0$ occurs if an only if $\tilde{D}^{-1}\left(k\left(\frac{R_1}{X} + \frac{R_2}{X}\right)\right) < u_2 + c_2 < \tilde{D}^{-1}\left(k\frac{R_1}{X}\right)$. In this regime, $g_1(R_1, R_2, X, u_1, u_2) = u_2 + c_2 - c_1$ and $g_2(R_1, R_2, X, u_1, u_2) = u_2$.

3. **Industry 1 produces at full capacity and industry 2 does not produce** when $u_1 + c_1 < p < u_2 + c_2$. The total productions are $Q_1 = kR_1$ and $Q_2 = 0$; matching offer and demand yields $P^*(R_1, R_2, X, u_1, u_2) = \tilde{D}^{-1}\left(k\frac{R_1}{X}\right)$. The inequality is equivalent to $u_2 + c_2 > \tilde{D}^{-1}\left(k\frac{R_1}{X}\right) > u_1 + c_1$. In this regime, $g_i(R_1, R_2, X, u_1, u_2) = \tilde{D}^{-1}\left(k\frac{R_1}{X}\right) - c_i$.

4. **Industry 1 has a positive but partial production and industry 2 does not produce** when $u_1 + c_1 = p < u_2 + c_2$. Then $Q_2 = 0$ and matching offer and demand yields $Q_1/X = \tilde{D}(c_1 + u_1)$. In this regime, $u_2 + c_2 > u_1 + c_1 > \tilde{D}^{-1}\left(k\frac{R_1}{X}\right)$, $g_1(R_1, R_2, X, u_1, u_2) = u_1$ and $g_2(R_1, R_2, X, u_1, u_2) = u_1 + c_1 - c_2$.

Summarizing, if $u1 + c1 < u_2 + c_2$, then the total productions are given by the continuous functions

$$Q_1^*(R_1, R_2, X, u_1, u_2) = X \min\left(\frac{kR_1}{X}, \tilde{D}(c_1 + u_1)\right) \tag{49}$$

$$Q_2^*(R_1, R_2, X, u_1, u_2) = X \max\left(0, \min\left(\frac{kR_2}{X}, \tilde{D}(c_2 + u_2) - \frac{kR_1}{X}\right)\right) \tag{50}$$

The optimal strategy of a production unit As above, the expected values u_i are obtained by optimizing on the controls, knowing the dynamics of R_1 and R_2:

$$\begin{aligned} &u_i(R_1, R_2, X) \\ =&(1 - r_i dt) \max_{\alpha_i > 0, 0 \le \beta_i \le k} \mathbb{E}\left(\begin{array}{l} (\beta_i g_i(R_1, R_2, X, u_1, u_2) - \alpha_i)dt \\ +(1 + \phi_i(\alpha_i)dt - \beta_i dt)u_i(R_1 + dR_1, R_2 + dR_2, X + dX) \end{array}\right) \end{aligned} \tag{51}$$

4.2 Partial differential equations

A first order expansion in (51) and the equilibrium relations $dR_i = (R_i\phi_i(\alpha_i^*) - Q_i^*(R_1, R_2, X, u_1, u_2))\, dt$, where α_i^* is the optimal value of α_i, yield the system of partial differential equations: for $i = 1, 2$,

$$\begin{aligned} 0 =& - r_i u_i + k\left(g_i(\cdot, u_1, u_2) - u_i\right) - Q_1^*(\cdot, u_1, u_2)\partial_{R_1} u_i - Q_2^*(\cdot, u_1, u_2)\partial_{R_2} u_i \\ &+ \partial_{R_i}\left(R_i \max_{\alpha_i}(u_i\phi_i(\alpha_i) - \alpha_i)\right) + \phi_j(\alpha_j^*)R_j\partial_{R_j} u_i + bX\partial_X u_i + \frac{1}{2}\sigma^2 X^2 \partial_{XX} u_i, \end{aligned} \tag{52}$$

where $j = 2$ (resp. $j = 1$) if $i = 1$ (resp. $i = 2$).

Homogeneity: reduced variables Observe that g_i and $\tilde{Q}_i^* = Q_i^*/X$ are functions of $y_1 = R_1/X$, $y_2 = R_2/X$, u_1 and u_2). It is natural to look for a solution of the form $u_i(R_1, R_2, X) =$

$v_i(y_1, y_2)$; this leads to the following system:

$$\begin{aligned}
0 =& - r_1 v_1 + k\left(g_1(y_1,y_2,v_1,v_2) - v_1\right) - \tilde{Q}_1^*(y_1,y_2,v_1,v_2)\partial_{y_1} v_1 - \tilde{Q}_2^*(y_1,y_2,v_1,v_2)\partial_{y_2} v_1 \\
&+ \partial_{y_1}\left(y_1 \max_{\alpha_1}(v_1\phi_1(\alpha_1) - \alpha_1)\right) + \phi_2(\alpha_2^*) y_2 \partial_{y_2} v_1 \\
&+ (\sigma^2 - b)(y_1\partial_{y_1} v_1 + y_2 \partial_{y_2} v_1) + \frac{1}{2}\sigma^2 (y_1^2 \partial^2_{y_1} v_1 + 2 y_1 y_2 \partial^2_{y_1 y_2} v_1 + y_2^2 \partial^2_{y_2} v_1) \\
0 =& - r_2 v_2 + k\left(g_2(y_1,y_2,v_1,v_2) - v_2\right) - \tilde{Q}_1^*(y_1,y_2,v_1,v_2)\partial_{y_1} v_2 - \tilde{Q}_2^*(y_1,y_2,v_1,v_2)\partial_{y_2} v_2 \\
&+ \partial_{y_2}\left(y_2 \max_{\alpha_2}(v_2\phi_2(\alpha_2) - \alpha_2)\right) + \phi_1(\alpha_1^*) y_1 \partial_{y_1} v_2 \\
&+ (\sigma^2 - b)(y_1\partial_{y_1} v_2 + y_2 \partial_{y_2} v_2) + \frac{1}{2}\sigma^2 (y_1^2 \partial^2_{y_1} v_2 + 2 y_1 y_2 \partial^2_{y_1 y_2} v_2 + y_2^2 \partial^2_{y_2} v_2)
\end{aligned} \tag{53}$$

A Hamilton-Jacobi equation In the case when $r_1 = r_2 = r$, consider the degenerate second order Hamilton-Jacobi equation:

$$(b - r)V - H(y_1, y_2, DV) - b\,(y_1\partial_{y_1} V + y_2 \partial_{y_2} V) + \frac{\sigma^2}{2}(y_1^2 \partial^2_{y_1} V + 2 y_1 y_2 \partial^2_{y_1 y_2} V + y_2^2 \partial^2_{y_2} V) = 0 \tag{54}$$

where $H(y_1, y_2, v_1, v_2) = H_1(y_1, y_2, v_1, v_2) + H_2(y_1, y_2, v_1, v_2)$ and

$$H_2(y_1, y_2, v_1, v_2) = -\sum_{i=1,2} y_i \max_{\alpha_i \ge 0} \left(\phi_i(\alpha_i) v_i - \alpha_i\right).$$

We give the expression of H_1 when $v_1 + c_1 < v_2 + c_2$. In the opposite case, it is enough to switch the indices $i = 1, 2$.

$$H_1(y_1, y_2, v_1, v_2) = \begin{cases}
k\left(y_1(v_1 + c_1) + y_2(v_2 + c_2) - (y_1 + y_2)\tilde{D}^{-1}(k(y_1 + y_2))\right) - \int_{D^{-1}(k(y_1+y_2))}^{M} \tilde{D}(z)dz & \text{if } D^{-1}(k(y_1 + y_2)) > v_2 + c_2, \\
k y_1 (v_1 + c_1 - v_2 - c_2) - \int_{v_2 + c_2}^{M} \tilde{D}(z)dz & \text{if } D^{-1}(k(y_1 + y_2)) < v_2 + c_2 < D^{-1}(k y_1), \\
k y_1 \left(v_1 + c_1 - \tilde{D}^{-1}(k y_1)\right) - \int_{D^{-1}(k y_1)}^{M} \tilde{D}(z)dz & \text{if } v_1 + c_1 < D^{-1}(k y_1) < v_2 + c_2, \\
-\int_{v_1 + c_1}^{M} \tilde{D}(z)dz & \text{if } D^{-1}(k y_1) < v_1 + c_1.
\end{cases}$$

The Hamiltonians H_1 and H_2 have the following monotonicity property:

$$(H_{k,p}(y,p) - H_{k,p}(z,q), p - q) - (H_{k,y}(y,p) - H_{k,y}(z,q), y - z) \le 0, \quad k = 1, 2. \tag{55}$$

If $r_1 = r_2 = r$ and V is a solution of (54) such that $\partial_1 V \geq 0$ and $\partial_2 V \geq 0$, then $(v_1, v_2) = DV$ is a solution to (53) with nonnegative components.
When D is given by (21), the Hamiltonian $H_1(\cdot, v_1, v_2)$ has the following form if $v_1 + c_1 < v_2 + c_2$:

$$H_1(y_1, y_2, v_1, v_2) = \begin{cases} ky_1(v_1 + c_1) + ky_2(v_2 + c_2) + \dfrac{s}{1-s} k^{1-\frac{1}{s}} (y_1 + y_2)^{1-\frac{1}{s}} \\ \qquad\qquad \text{if } \left(k\left(y_1 + y_2\right)\right)^{-\frac{1}{s}} > v_2 + c_2, \\ ky_1(v_1 + c_1 - v_2 - c_2) + \dfrac{1}{1-s} (v_2 + c_2)^{1-s} \\ \qquad\qquad \text{if } \left(k\left(y_1 + y_2\right)\right)^{-\frac{1}{s}} < v_2 + c_2 < (ky_1)^{-\frac{1}{s}}, \\ ky_1\left(v_1 + c_1\right) + \dfrac{s}{1-s} k^{1-\frac{1}{s}} y_1^{1-\frac{1}{s}} \\ \qquad\qquad \text{if } v_1 + c_1 < (ky_1)^{-\frac{1}{s}} < v_2 + c_2, \\ \dfrac{1}{1-s} (c_1 + v_1)^{1-s} \\ \qquad\qquad \text{if } (ky_1)^{-\frac{1}{s}} < v_1 + c_1. \end{cases} \tag{56}$$

5 Tuning the parameters

We consider the model described in § 2.1, assuming that the function ϕ describing the efficiency of prospection is of the form

$$\phi(\alpha) = C\sqrt{\alpha}, \tag{57}$$

and that the demand function if of the form $D(X, p) = Xp^{-s}$. The model therefore depends on a set $\mathcal{S}$ of seven parameters, namely

- the interest rate r
- the growth rate b and the volatility σ of the process X_t
- the production cost c and the production capacity k
- the parameter C in (57)
- the exponent s in the demand function.

From (1), (10), the reduced variable $y_t = R_t/X_t$ satisfies the stochastic differential equation:

$$dy_t = \Psi_{\mathcal{S}}(y_t)dt - \sigma y_t dW_t, \tag{58}$$

with the drift given by

$$\Psi_{\mathcal{S}}(y) = \left(Cv(y) - \min\left(k, \frac{1}{y(c + v(y))^s}\right) + \sigma^2 - b \right) y \tag{59}$$

The model must be calibrated in order to fit the data, which is, for a given material (e.g. copper, zinc, nickel, cobalt), the series of the prices every month in the last 40 years. There are thus 480 observed prices $(p_i)_{i=1,\dots,480}$, see for example Figure 1 where the price of copper is plotted. The time interval between two observations is $\Delta t = 1/12$. Let $t_i = i\Delta t$ be the date of the i^{th} observation.

Let $v_{\mathcal{S}}$ be the solution to (22) when the parameter set if $\mathcal{S}$. Knowing $\mathcal{S}$ and $v_{\mathcal{S}}$, we can map any observed prices p_i to a value y_i by inverting $p_i = \max\left((ky_i)^{-1/s}, v_{\mathcal{S}}(y_i) + c\right)$, see (4).
The parameters estimation consists of maximizing the likelihood of the observations. This amounts to minimizing

$$J(\mathcal{S}) = \frac{1}{2}\sum_i \left(\frac{y_{i+1} - y_i - \Delta t \Psi_{\mathcal{S}}(y_i)}{\sigma y_i \sqrt{\Delta t}}\right)^2 + \sum_i \ln(\sigma y_i), \tag{60}$$

given a positive parameter ϵ. We also impose some constraints on the parameters. The constrained minimization of J is done using Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm.

For copper, we found that the set of parameters: $r = 0.18$, $b = 0.01$, $c = 0.2$ $k = 0.29$, $C = 1$, $\sigma = 0.20$ and $s = 0.8$ permits to fit the data in a rather satisfactory manner. In Figure 2, we plot both the distribution of the observed prices (dotted line) and the distribution of the prices predicted by the model with the above parameters. In both the observed and predicted distributions, there is narrow peak correponding to small prices at $p \approx 0.6$ and a bump for larger prices: the narrow peak corresponds to periods when the demand is low, during which $p = c + u(R, X)$ and $Q(R, X) = X/(c + u(R, X))^s$. The bump corresponds to periods when the demand is high, during which $p > c + u(R, X)$ and $Q(R, X) = kR$.
In Figure 3, we plot the optimal price as a function of R/X. The two different regimes discussed in § 2.1.2 appear clearly: in the first regime, corresponding to small values of R/X (the bump in the price distribution in Figure 2) the industry produces at full capacity. In the second regime, corresponding to higher values of R/X, (the peaks in the price distribution in Figure 2), the price is low, and the industry has a partial production.

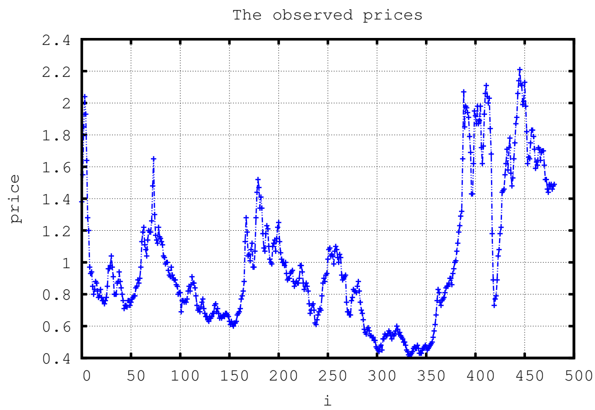


Figure 1: The observed prices of copper

We have carried the same program for several materials: in Figure 4, we compare the historical and predicted distributions of prices for zinc, with the following set of parameters: $r = 0.2$, $b = 0.01$, $c = 0.2$ $k = 0.35$, $C = 1.05$, $\sigma = 0.23$ and $s = 0.92$

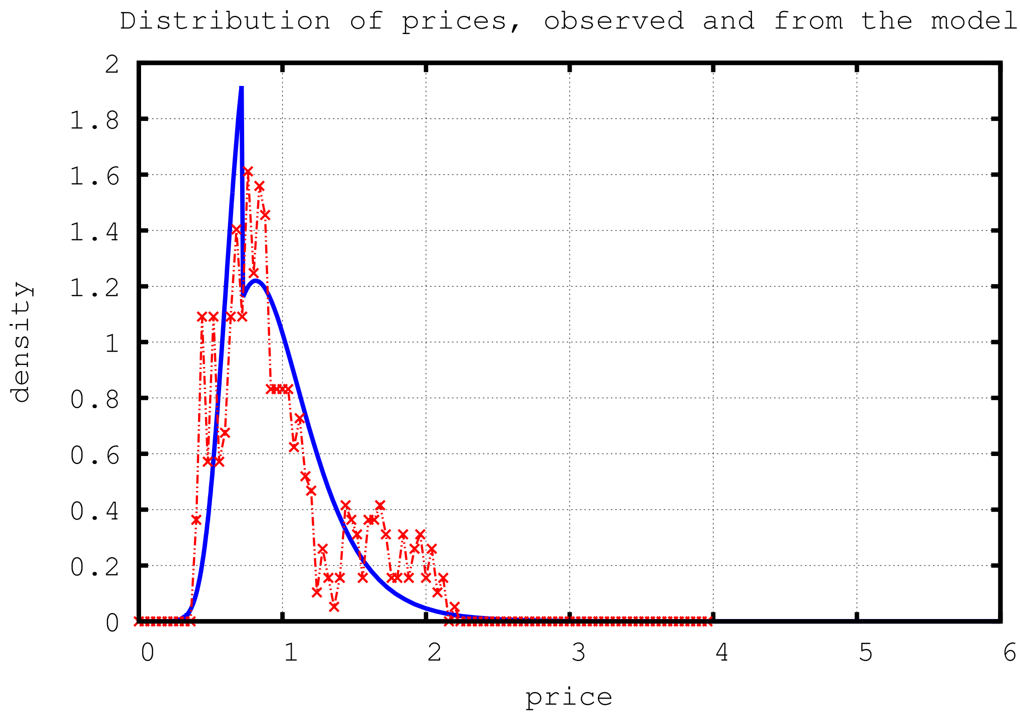


Figure 2: Copper: the distribution of the prices (observed and computed from the model)

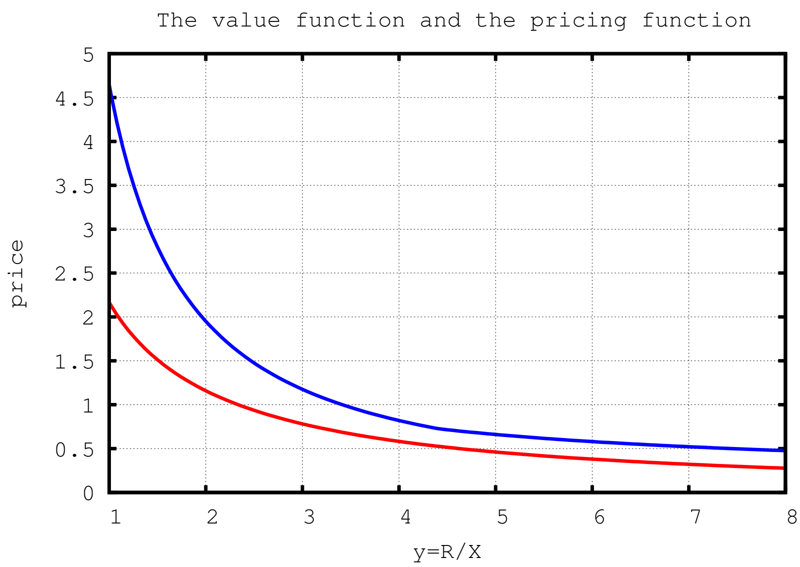


Figure 3: Copper: in red : the solution v of (12); in blue: the price P^* predicted by the model as a function of $y = R/X$.

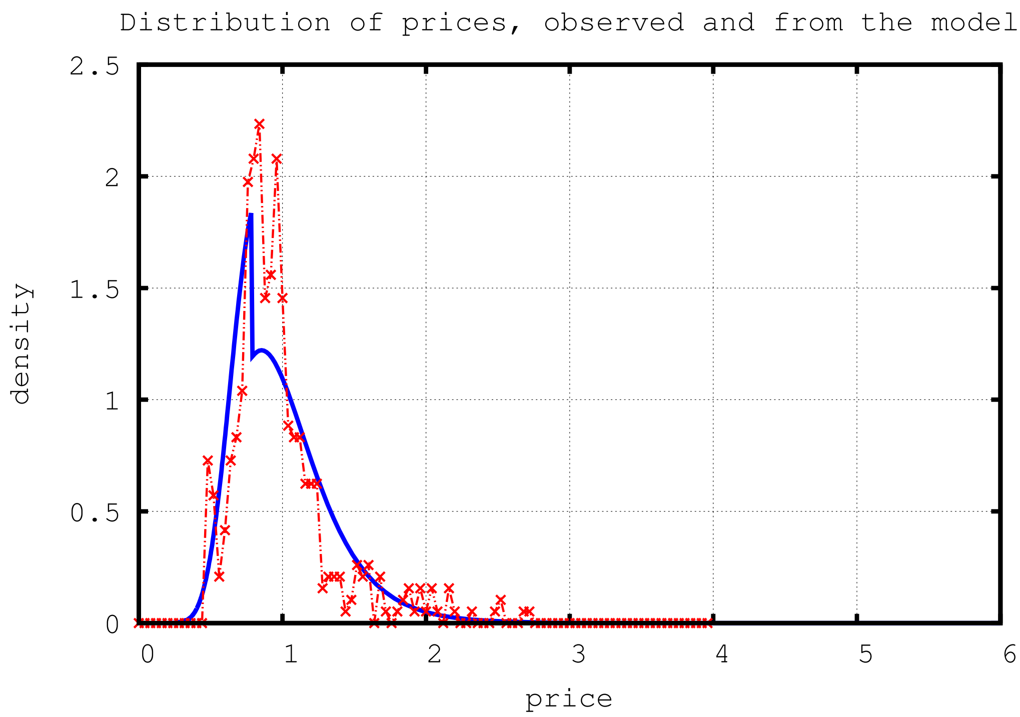


Figure 4: Zinc: the distribution of the prices (observed and computed from the model)

6 Numerical simulations of the closed industry with two technologies

6.1 Case 1

We consider the model presented in § 4 with

$$\tilde{D}(p) = p^{-s}, \quad \phi_1(\alpha) = \phi_2(\alpha) = 0.895\sqrt{\alpha},$$

with the following parameters

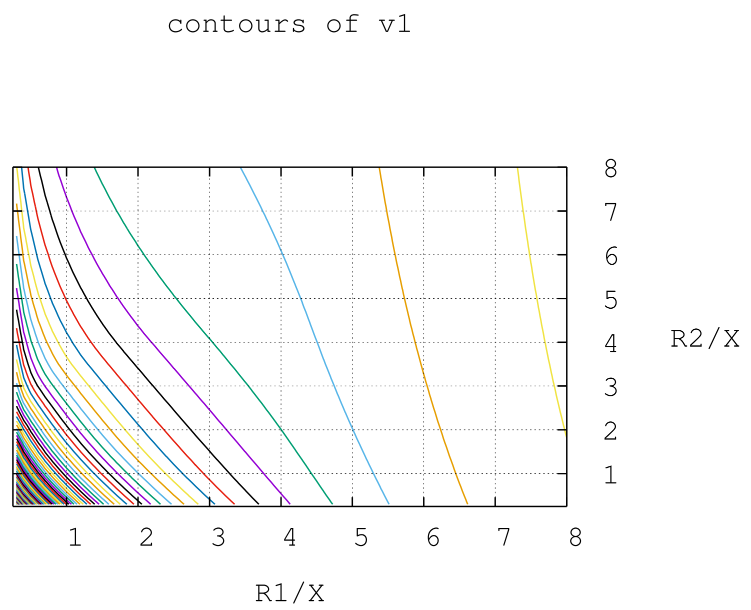
$$r_1 = r_2 = 0.18; \; c_1 = 0.35; \; c_2 = 0.6;$$
$$k = 0.3; \; s = 0.6; \; \sigma = 0.15; \; b = 0.04;$$

In this model, the cost of production of industry 1 is smaller than that of industry 2: $c_1 < c_2$, whereas the investments into prospection are equally efficient.

On Figure 5, we plot the contours of v_1 and v_2 as functions of $y_1 = R_1/X$ and $y_1 = R_2/X$. Note that both v_1 and v_2 blow up at $(0,0)$.

On Figure 6, we plot the rescaled productions $\tilde{Q}_1 = \frac{Q_1}{X}$ and $\tilde{Q}_2 = \frac{Q_2}{X}$ as functions of $y_1 = R_1/X$ and $y_2 = R_2/X$. We plot the contours of the same functions on Figure 7. Note the region near the y_2 axis in which the production of industry 1 is zero.

On Figure 9, we plot the different zones corresponding to the different regimes of the Hamiltonian. The zones numbered from 0 to 3 correspond to the case when $v_1 + c_1 \leq v_2 + c_2$ and to the four successive regimes in the definition of H_1 in (56). The zones numbered from 4 to 7 correspond to the case when $v_2 + c_2 \leq v_1 + c_1$ and to the four related regimes in the symmetrized version of (56). We see that all the regimes are present except the first one in the case $v_1 + c_1 \leq v_2 + c_2$, i.e. there are seven different zones.



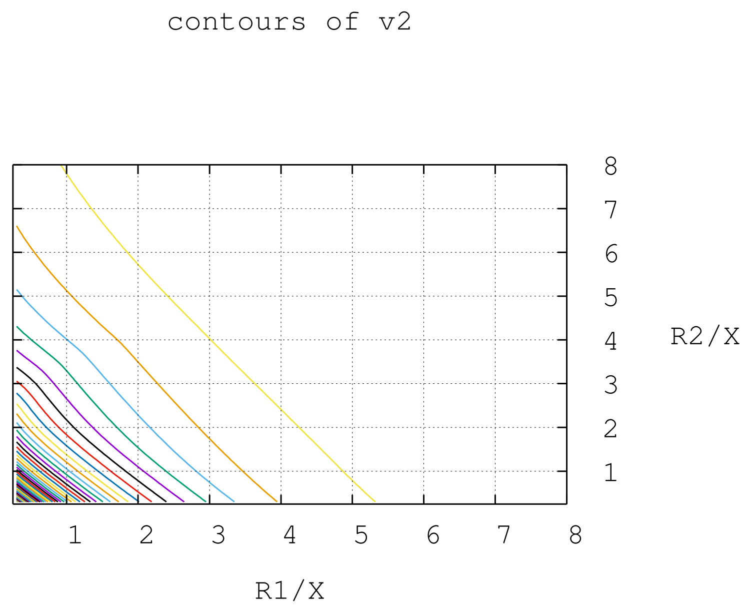
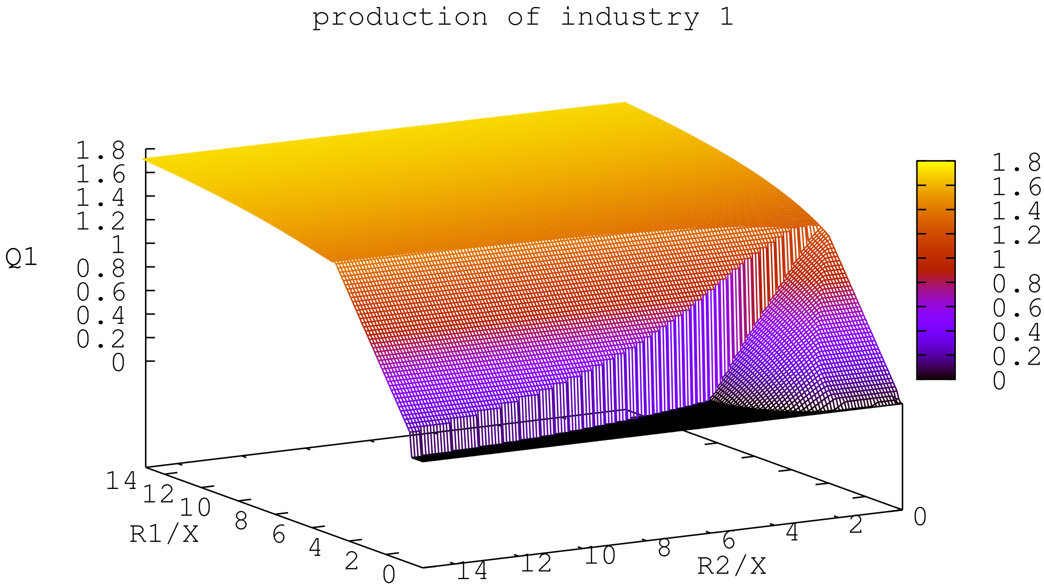


Figure 5: The contours of v_1 and v_2



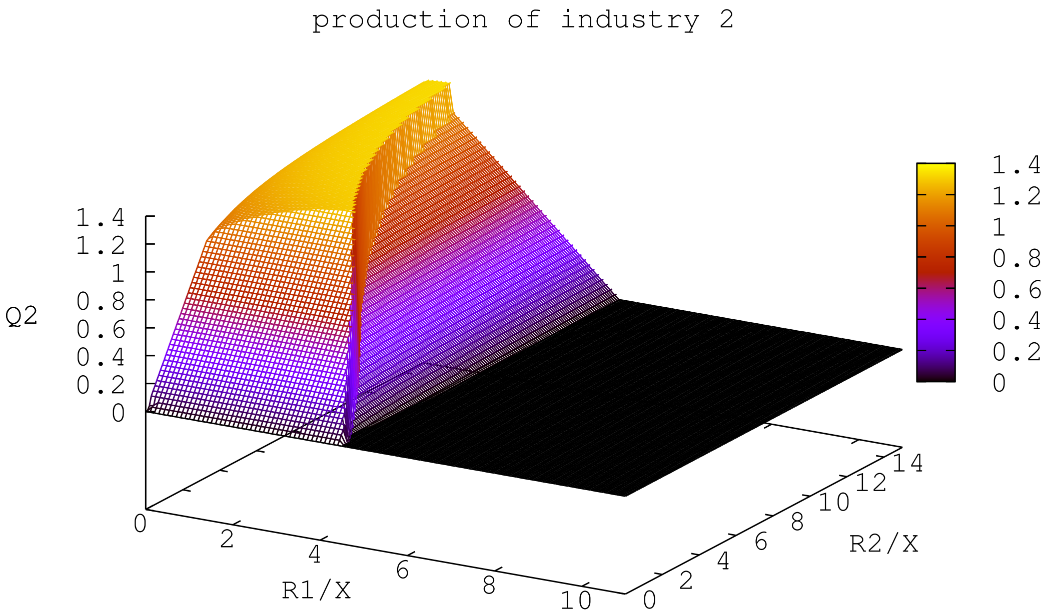
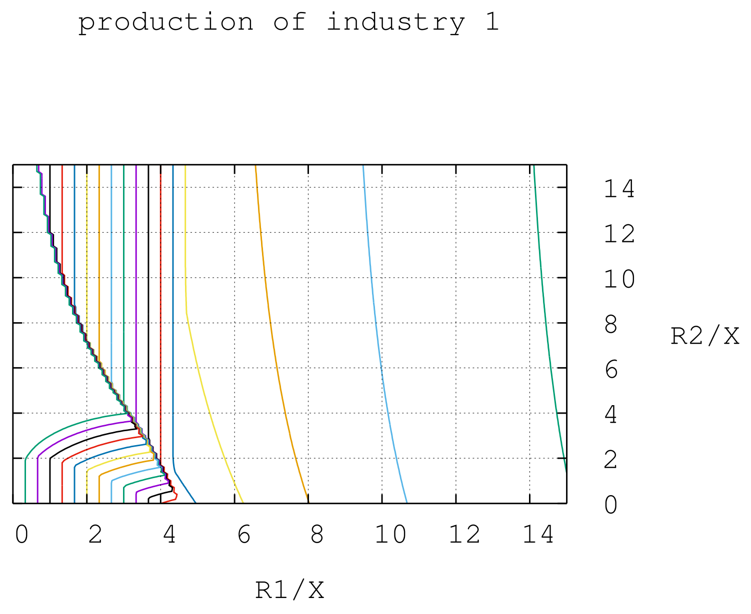


Figure 6: The productions $\tilde{Q}_1 = \frac{Q_1}{X}$ and $\tilde{Q}_2 = \frac{Q_2}{X}$



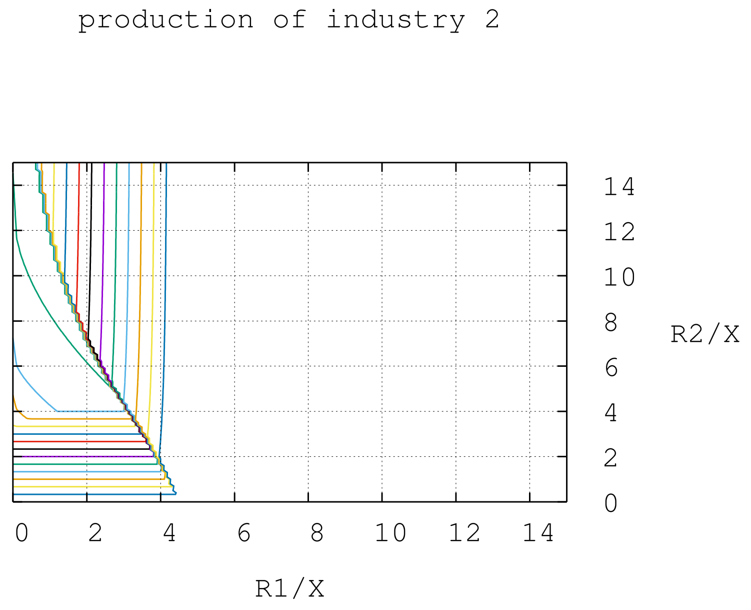


Figure 7: The contours of the productions $\tilde{Q}_1 = \frac{Q_1}{X}$ and $\tilde{Q}_2 = \frac{Q_2}{X}$

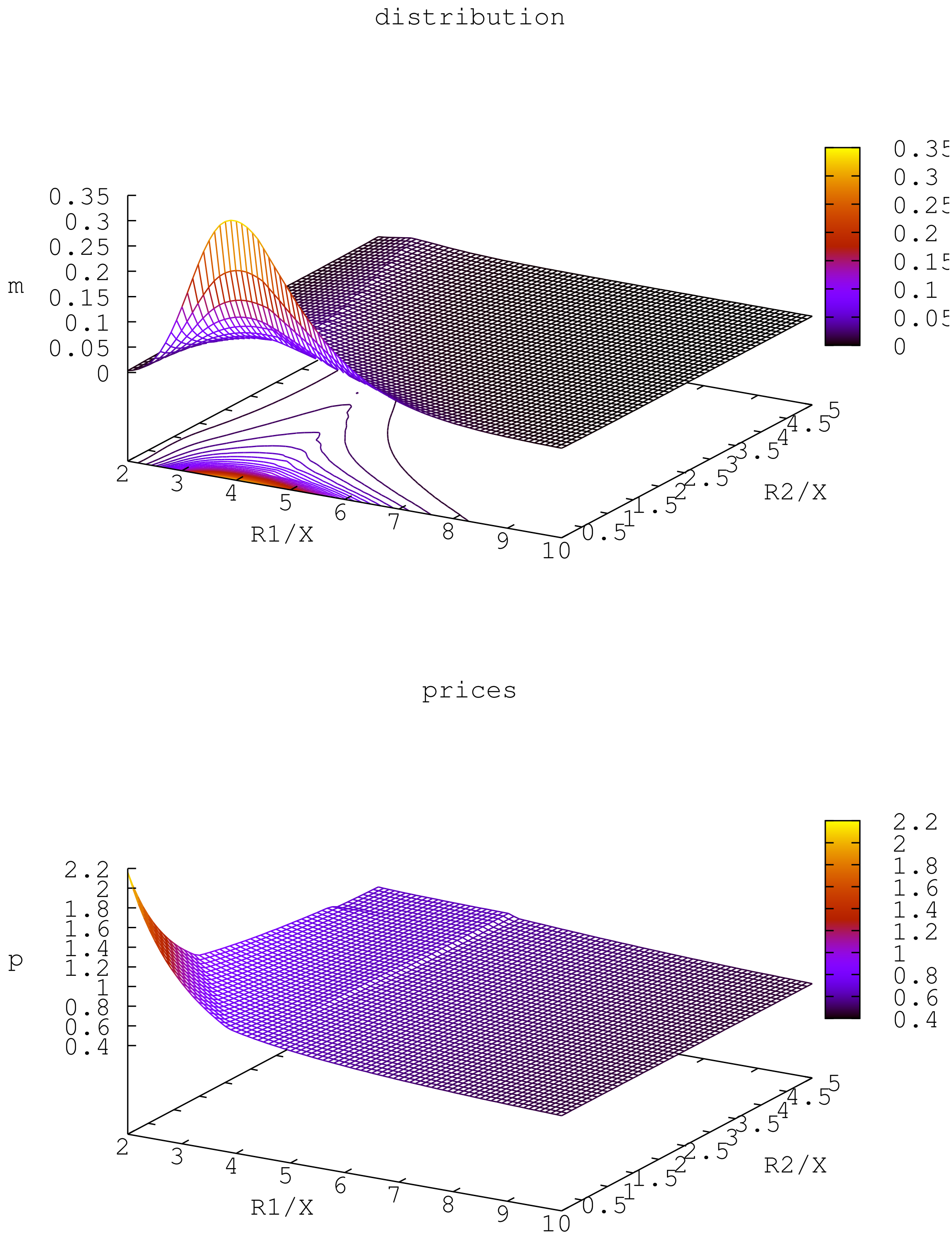


Figure 8: The distributions of the agents and the prices

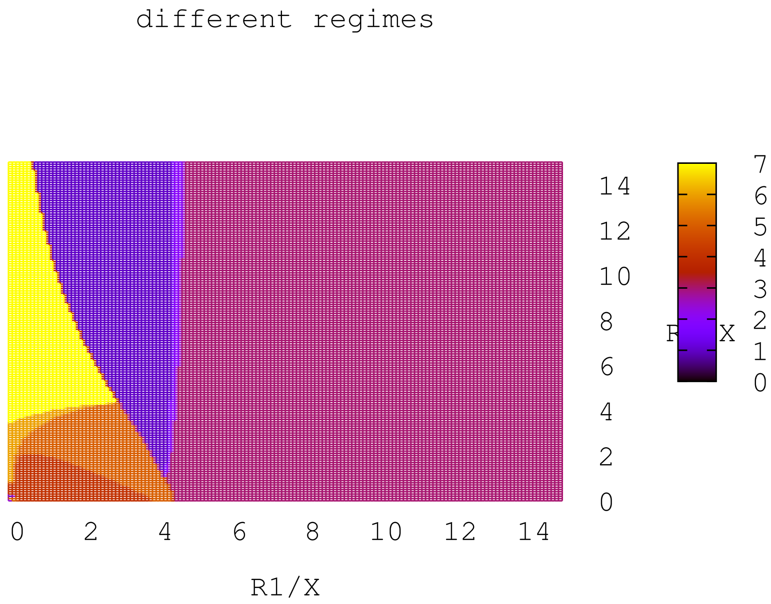


Figure 9: There are seven different regimes for the Hamiltonians (see § 4.1): all the possible regimes are present except the first regime in the case when $v_1 + c_1 \leq v_2 + c_2$.

6.2 Case 2

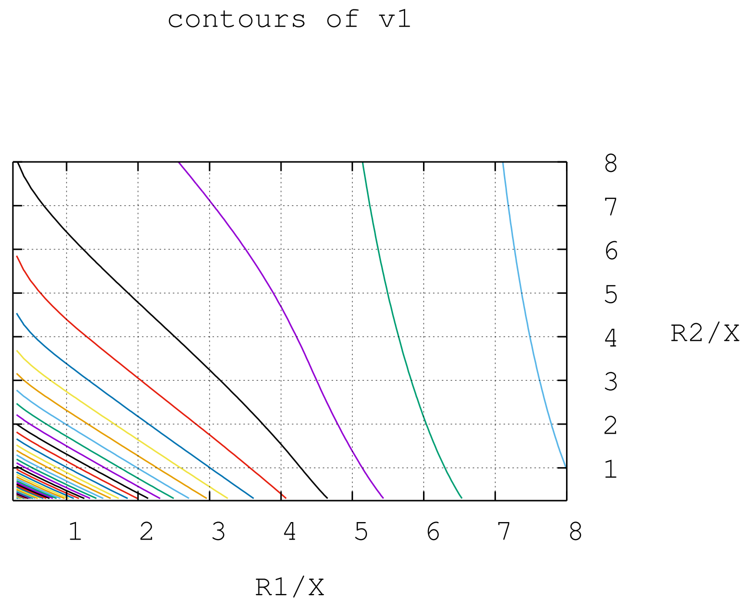
We keep the same parameters as above except that

$$\phi_1(\alpha) = 0.895\sqrt{\alpha}, \quad \phi_2(\alpha) = 1.183\sqrt{\alpha},$$

In this model, the cost of production of industry 1 is smaller than that of industry 2: $c_1 < c_2$, whereas the investments into prospection are more efficient for industry 2.

On Figure 10, we plot the contours of v_1 and v_2 as functions of $y_1 = R_1/X$ and $y_1 = R_2/X$. On Figure 11, we plot the rescaled productions $\tilde{Q}_1 = \frac{Q_1}{X}$ and $\tilde{Q}_2 = \frac{Q_2}{X}$ as functions of $y_1 = R_1/X$ and $y_2 = R_2/X$. We plot the contours of the same functions on Figure 12.

On Figure 14, we plot eight different zones corresponding to the different regimes of the Hamiltonian. All the eight possible regimes are present.



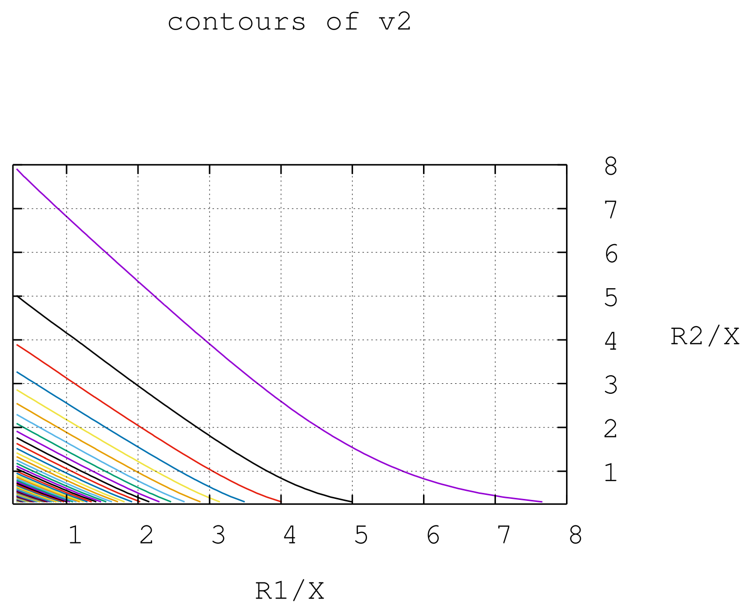
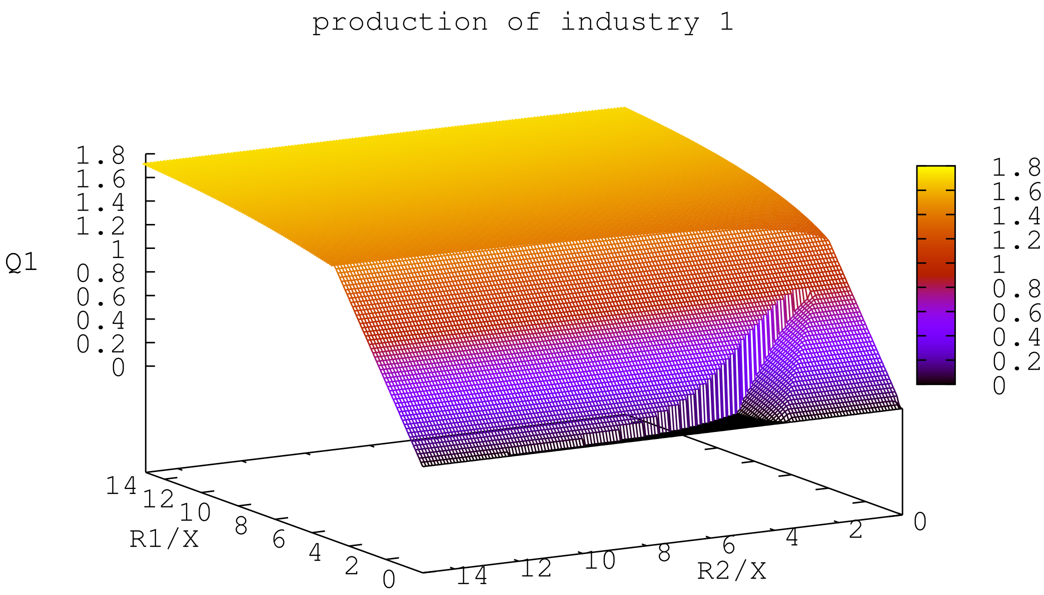


Figure 10: The contours of v_1 and v_2



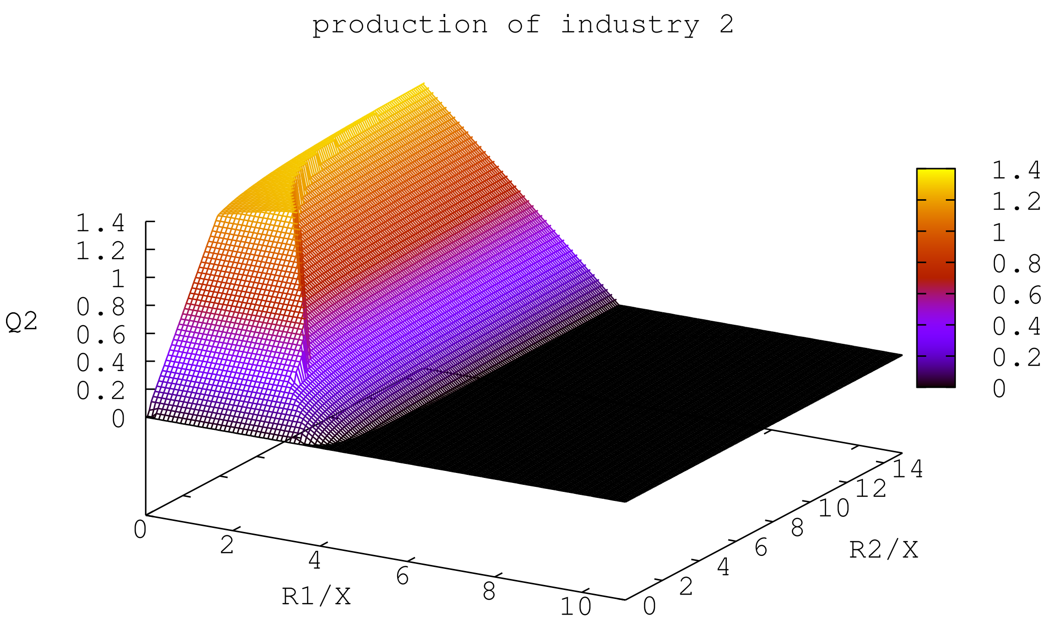
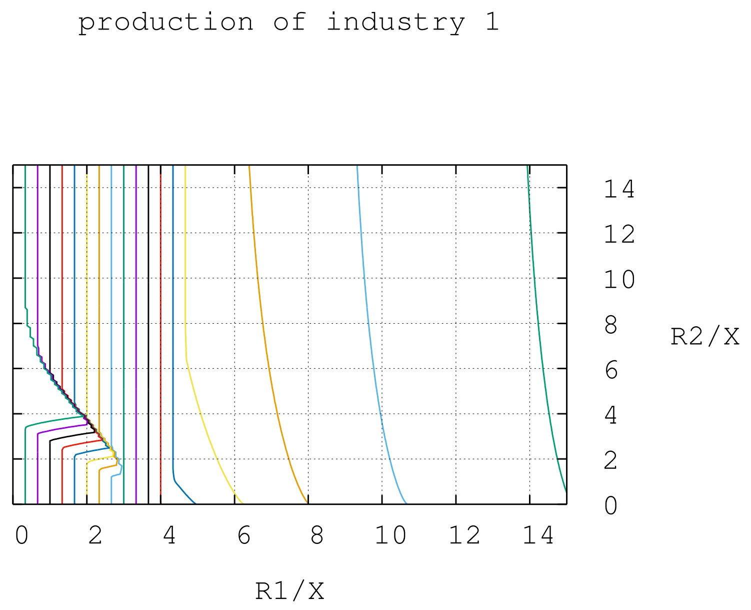


Figure 11: The productions $\tilde{Q}_1 = \frac{Q_1}{X}$ and $\tilde{Q}_2 = \frac{Q_2}{X}$



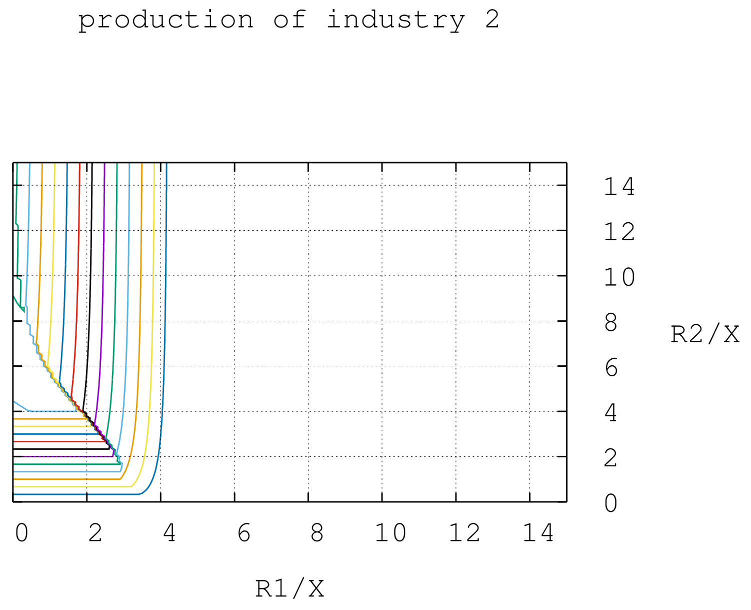


Figure 12: The contours of the productions $\tilde{Q}_1 = \frac{Q_1}{X}$ and $\tilde{Q}_2 = \frac{Q_2}{X}$

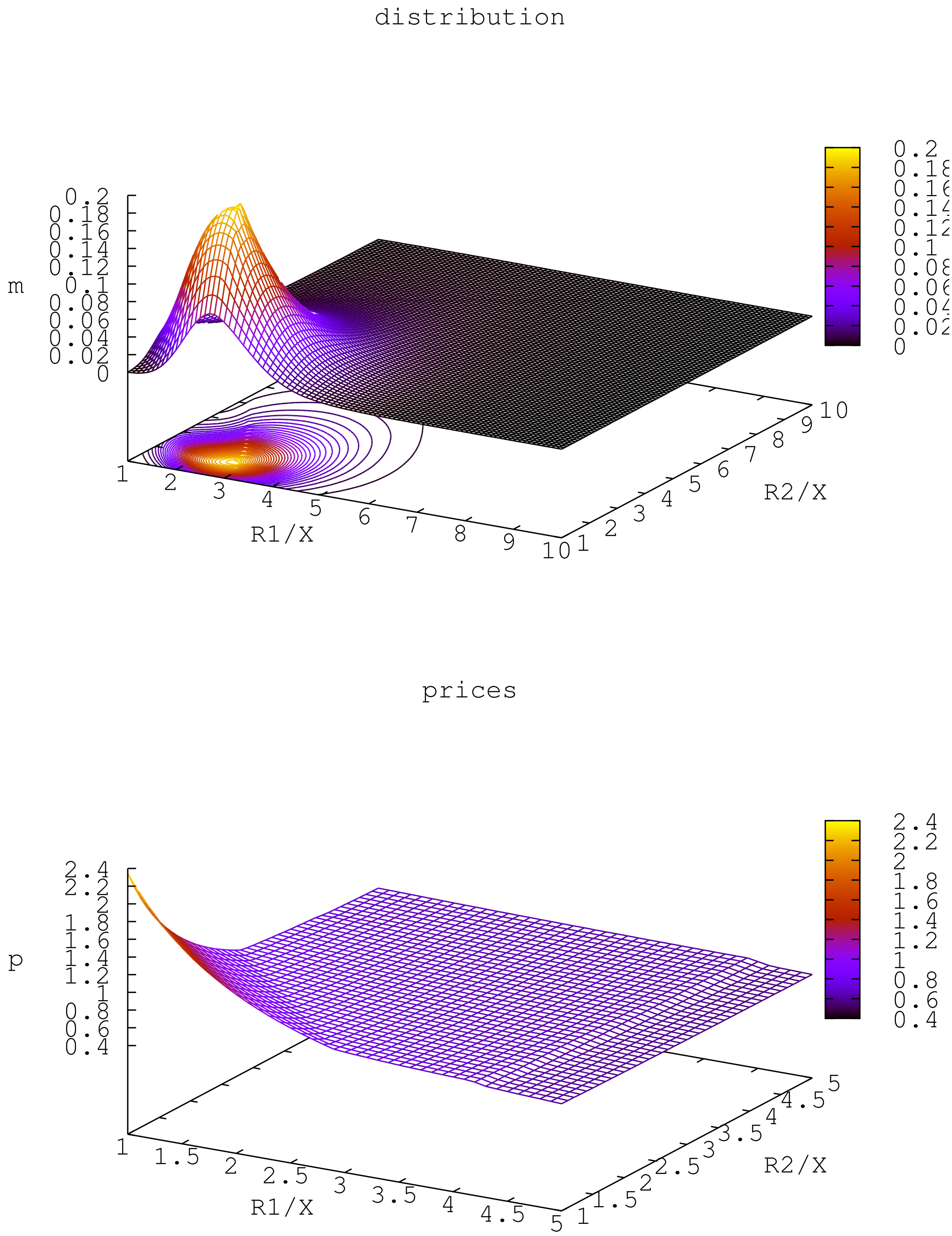


Figure 13: The distributions of the agents and the prices

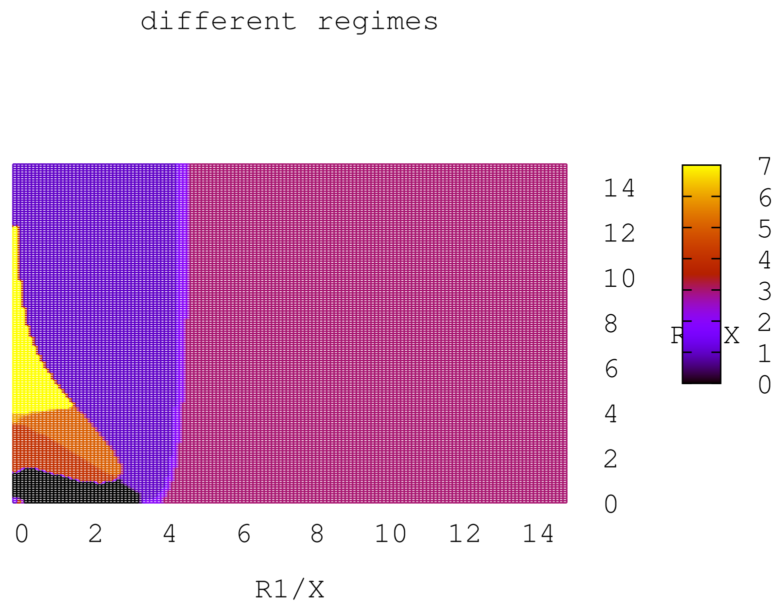


Figure 14: There are eight different regimes for the Hamiltonians (see § 4.1)

A A mean field games approach to Lucas-Prescott benevolent planner model

The present mining industry model is reminiscent of the celebrated Lucas-Prescott model, see [11]: let us recall the latter and propose its interpretation in terms of mean field games.

A.1 The framework

Lucas and Prescott consider a market of production units. The size of a given unit, i.e. the capital owned by the producer is k_t. Each producer can invest in order to improve the production capacity, and therefore increase its capital: the flux of capital generated during dt by an invested flux of $z_t dt$ is $k_t \Phi(z_t/k_t)dt$, where the nondecreasing and concave function ϕ measures the impact of the investment: therefore

$$dk_t = k_t \Phi(z_t/k_t)dt.$$

The price of a unit of capital is fixed by a pricing function:

$$p_t = P(K_t, X_t),$$

where K_t is the aggregate capital and X_t is an exogeneous parameter standing for the global state of the economy. This parameter is driven by a diffusion process:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,$$

hence it represents an aggregate risk, common to all production units.
Each individual firm solves the optimal control problem: maximiz e

$$u = \max_{z_t \geq 0} \mathbb{E}\left(\int_0^\infty e^{-rt}(p_t k_t - z_t)dt\right).$$

A.2 The approach via mean field games

Similarly as in § 2, it is convenient to split the production units in such a way that any production unit corresponds to a unit of capital. Then the aggregate capital K_t is the number of such production units. The value u of a unit of capital can be expressed as a function of K and X: $u = u(K, X)$. From the dynamic programming principle, u satisfies

$$u(K, X) = (1 - rdt) \max_{\alpha \geq 0} \mathbb{E}\Big((1 + \phi(\alpha)dt)u(K + dK, X + dX) + (P(K, X) - \alpha)dt\Big). \tag{61}$$

Note the similarity with (7). A first order expansion yields

$$\begin{aligned}&\max_{\alpha \geq 0}(\phi(\alpha)u(K, X) - \alpha) + \partial_K u(K, X)\frac{dK}{dt} + \mu(X)\partial_X u(K, X) + \frac{\sigma^2(X)}{2}\partial_X^2 u(K, X) + P(K, X)\\ &= ru(K, X),\end{aligned}$$

where the optimal control α^* is such that $\phi'(\alpha^*)u(K, X) = 1$. Then, at the mean field game equilibrium, $\frac{dK}{dt} = \phi(\alpha^*)K$. Finally, we obtain the partial differential equation:

$$\begin{aligned}&\max_{\alpha \geq 0}(\phi(\alpha)u(K, X) - \alpha) + K\phi(\alpha^*)\partial_K u(K, X) + \mu(X)\partial_X u(K, X) + \frac{\sigma^2(X)}{2}\partial_X^2 u(K, X) + P(K, X)\\ &= ru(K, X),\end{aligned}$$

or in an equivalent manner

$$\partial_K\left(K \max_{\alpha \geq 0}(\phi(\alpha)u(K, X) - \alpha)\right) + \mu(X)\partial_X u(K, X) + \frac{\sigma^2(X)}{2}\partial_X^2 u(K, X) + P(K, X) = ru(K, X). \tag{62}$$

A.3 From (62) to a Hamilton-Jacobi equation

The aim is to find a Hamilton-Jacobi equation of the form

$$rV(K, X) = H(K, X, \partial_K V, \partial_X V) + a(X)\partial_X^2 V, \tag{63}$$

in such a way that if V is a solution to (63), then $u = \partial_K V$ is a solution to (62). Differentiating (63) with respect to K,

$$\begin{aligned}r\partial_K V =&\partial_K H(K, X, \partial_K V, \partial_X V)\\ &+ \partial_3 H(K, X, \partial_K V, \partial_X V)\partial_K^2 V + \partial_4 H(K, X, \partial_K V, \partial_X V)\partial_{KX}^2 V + a(X)\partial_{XXK}^3 V,\end{aligned}$$

with self-explanatory notations. If $u = \partial_K V$, this yields

$$ru = \partial_K H(K, X, u, \partial_X V) + \partial_3 H(K, X, u, \partial_X V)\partial_K u + \partial_4 H(K, X, u, \partial_X V)\partial_X u + a(X)\partial_X^2 u$$

Identifying, we obtain that $a(X) = \sigma^2(X)/2$, $\partial_4 H(K, X, u, \partial_X V) = \mu(X)$, $\partial_3 H(K, X, u, \partial_X V) = K\Phi(\alpha^*)$ and $\partial_K H(K, X, u, \partial_X V) = P(K, X) + \max_{\alpha \geq 0}(\phi(\alpha)u - \alpha)$. Consider a surplus function $s(K, X)$ such that $\partial_K s(K, X) = P(K, X)$; we see that a good candidate for the Hamiltonian is

$$H(K, X, u, z) = K \max_{\alpha \geq 0}(\phi(\alpha)u - \alpha) + \mu(X)z + s(K, X).$$

and (63) becomes

$$rV(K,X) = K\max_{\alpha\geq 0}(\phi(\alpha)\partial_K V - \alpha) + \mu(X)\partial_X V + \frac{\sigma^2(X)}{2}\partial_X^2 V + s(K,X), \tag{64}$$

which plays the same role as (18), except that no reduced variable as been used in (64). Note that differentiating (64) with respect to X and setting $w(K,X) = \partial_X V$ yields another partial equation:

$$K(\phi(\alpha^*)\partial_K w(K,X) + \partial_X(\mu w) + \partial_X\left(\frac{\sigma^2}{2}\partial_X w\right) + \partial_X s(K,X) = rw(K,X). \tag{65}$$

A.4 Link of (64) with Lucas-Prescott benevolent planner problem

Hamilton-Jacobi equation (64), which has been found via mean field game theory, is also satisfied by the value function of Lucas-Prescott benevolent planner problem. Indeed, Lucas-Prescott benevolent planner problem is as follows:

$$V(K,X) = \max_{\alpha_t}\mathbb{E}\left(\int_0^\infty e^{-rt}(s(K_t,X_t) - \alpha_t K_t)dt\right), \tag{66}$$

subject to

$$dK_t = K_t\Phi(\alpha_t)dt, \qquad K_0 = K \tag{67}$$

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \qquad X_0 = X, \tag{68}$$

and from the dynamic programming principle, V solves (64).

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